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Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant

Edward Bierstone, Pierre D. Milman*

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 1A1
(e-mail: bierston@math.toronto.edu; milman@math.toronto.edu)

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Dedicated to Heisuke Hironaka

Summary. This article contains an elementary constructive proof of resolution of singularities in characteristic zero. Our proof applies in particular to schemes of finite type and to analytic spaces (so we recover the great theorems of Hironaka). We introduce a discrete local invariant $\mathrm{inv}_X(a)$ whose maximum locus determines a smooth centre of blowing up, leading to desingularization. To define inv_X, we need only to work with a category of local-ringed spaces $X = (|X|, \mathcal{O}_X)$ satisfying certain natural conditions. If $a \in |X|$, then $\mathrm{inv}_X(a)$ depends only on $\widehat{\mathcal{O}}_{X,a}$. More generally, inv_X is defined inductively after any sequence of blowings-up whose centres have only normal crossings with respect to the exceptional divisors and lie in the constant loci of $\mathrm{inv}_X(\cdot)$. The paper is self-contained and includes detailed examples. One of our goals is that the reader understand the desingularization theorem, rather than simply "know" it is true.

Contents

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Chapter I. Introduction

This article contains an elementary constructive proof of resolution of singularities in characteristic zero. $\underline{k}$ will denote a field of characteristic zero throughout the paper. Our proof applies to a scheme X of finite type over $\underline{k}$, or to an analytic space X over $\underline{k}$ (in the case that $\underline{k}$ has a complete valuation); we recover, in particular, the great theorems of Hironaka [H1,2], [AHV1,2]. But our work neither was conceived nor is written in the modern language of algebraic geometry. We introduce a discrete local invariant $\mathrm{inv}_X(a)$ whose maximum locus determines a centre of blowing up, leading to desingularization. To define inv_X, we need only to work with a category $\mathscr{A}$ of local-ringed spaces $X = (|X|, \mathscr{O}_X)$ over $\underline{k}$ satisfying certain mild conditions (Remark 1.5), although further restrictions on $\mathscr{A}$ are needed for global resolution of singularities.

If $a \in |X|$, then $\mathrm{inv}_X(a)$ depends only on the completed local ring $\widehat{\mathscr{O}}_{X,a}$. In general, inv_X is defined recursively over a sequence of blowings-up whose centres have only normal crossings with respect to the exceptional divisors and lie in the constant loci of inv_X. (See (1.2).) inv_X takes only finitely many maximum values (at least locally). Moreover, its maximum locus has only normal crossings and each of its local components extends to a global smooth subspace, justifying the philosophy that "a sufficiently good local choice [of centre of blowing-up] should globalize automatically" [BM4].

(0.1) Our desingularization algorithm applies to the following classes of spaces:

(1) *Algebraic.* Schemes of finite type over $\underline{k}$ (cf. [H1]). Algebraic spaces over $\underline{k}$ (in the sense of Artin [Ar], Knutson [Kn]). Restrictions of schemes X of finite type over $\underline{k}$ to their $\underline{k}$-rational points $|X|_{\underline{k}}$. (Such spaces might be the natural object of study when our main interest lies in the $\underline{k}$-rational points; e.g., for real algebraic varieties.)

(2) *Analytic.* Real or complex analytic spaces (cf. [H2], [AHV1,2]). p-adic analytic spaces in the sense of Serre [Se] or Berkovich [Ber].

(3) "Quasianalytic hypersurfaces", defined by sheaves of principal ideals, each locally generated by a single quasianalytic function, on quasianalytic manifolds in the sense of E.M. Dyn'kin [D] (a class intermediate between analytic and C^∞).

In each of the classes of (0.1), a space X is locally a subspace of a manifold, or smooth space, $M = (|M|, \mathcal{O}_M)$. For the purpose of global desingularization, a key property of our category of spaces $\mathcal{A}$ is the following:

(0.2) A manifold M in $\mathcal{A}$ can be covered by "regular coordinate charts" U: the coordinates $(x_1, \ldots, x_n)$ on U are "regular functions" on U (i.e., each $x_i \in \mathcal{O}_M(U)$) and the partial derivatives $\partial^{|\alpha|}/\partial x^\alpha = \partial^{\alpha_1+\cdots+\alpha_n}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ make sense as transformations $\mathcal{O}_M(U) \to \mathcal{O}_M(U)$. Moreover, for each $a \in U$, there is an injective "Taylor series homomorphism" $T_a\colon \mathcal{O}_{M,a} \to \mathbb{F}_a[[X]] = \mathbb{F}_a[[X_1, \ldots, X_n]]$, where $\mathbb{F}_a$ denotes the residue field $\mathcal{O}_{X,a}/\underline{m}_{X,a}$, such that T_a induces an isomorphism $\widehat{\mathcal{O}}_{M,a} \xrightarrow{\cong} \mathbb{F}_a[[X]]$ and T_a commutes with differentiation: $T_a \circ (\partial^{|\alpha|}/\partial x^\alpha) = (\partial^{|\alpha|}/\partial X^\alpha) \circ T_a$, for all $\alpha \in \mathbb{N}^n$. ($\underline{m}_{X,a}$ denotes the maximal ideal of $\mathcal{O}_{X,a}$.)

In Sect. 3 below, we will give a more precise list of the properties of our category of spaces $\mathcal{A}$ that we use to prove global desingularization. As an application of our theorem, we show that desingularization (in the hypersurface case) implies Łojasiewicz's inequalities (Sect. 2). (These inequalities seem to be new for quasianalytic functions in dimension > 2.) Our invariant can also be applied to desingularization of "quasi-Noetherian spaces", generalizing Pfaffian varieties in the sense of Khovanskii.

Our results here were announced in [BM6], and extend techniques introduced in [BM3] and [BM4]. When we began thinking about this subject more than fifteen years ago, we were motivated by a simple desire to understand how to resolve singularities. One of our goals is that the reader understand the desingularization theorem, rather than simply "know" it is true. We believe that the invariant inv_X is of interest as a local measure of singularity, beyond desingularization itself. Significant general features of this work in comparison to previous published treatments include: (i) Our desingularization theorems are canonical (cf. Remark 1.16 ff. and Sect. 13). (ii) We isolate local properties of an invariant (Theorem 1.14) from which globalization is automatic (Remark 1.15 ff. or 10.3 ff.). (iii) Our proof in the case of a hypersurface (a space defined locally by a single equation) does not involve passing to higher codimension (as in [H1]).

Our notion of "presentation" (Sects. 1,4) has much in common with Hironaka's idea of "strong local equivalence" of idealistic exponents [H3], although the analogy with [H3] does not seem to go beyond "presentation of the Hilbert-Samuel function" (in the language of this article). Our proof of resolution of singularities combines the uniformization algorithm of [BM3, Sect. 4] with the way we structure the notion of presentation. Essential points include the way we encode the history of the resolution process, originating in [BM3, Sect. 4] (and used in a similar way in [V1]), and the introduction of "exceptional blowings-up" (Sect. 4). We use these notions to develop a calculus for resolving singularities (cf. Example 2.1). The idea is to introduce new variables into the equations of a variety (by taking products with lines and test blowings-up) in a way that isolates invariant "blocks" representing important geometric features (cf. 1.11, 3.25). The idea of using test blowings-up to distinguish invariants occurs in [H3] (as Villamayor pointed out to us in 1990), as well as in [Ab2] (cf. [Li]).

Our "presentation" is by "regular functions" (functions in the class considered). Lack of such a presentation is a source of difficulty in previous treatments of desingularization. Regularity makes our algorithm work in the general context of the paper, for naive reasons. Ideas important to the general case (Ch. III) are isolation of the division properties of a local ideal that survive locally throughout the Samuel stratum (Sect. 7.2), an elementary stabilization theorem for homogeneous polynomials (Sect. 8), and an "implicit differention" property (referred to in 1.19, but realized a little differently in Sect. 9).

Notwithstanding the comparisons made above, we admit having not fully understood any other proof of desingularization. At the same time, our debt to the philosophy of Hironaka is greater than can be measured by precise references to his results. For a guide to the literature before 1976, we recommend Hironaka's bibliographical commentary in [H3]; our references include the more recent publications that we know of. Mark Spivakovsky has announced a proof of desingularization of arbitrary excellent schemes [Sp]. (A weaker theorem for any characteristic has been proved by de Jong [dJ].)

We are happy to thank Christof Waltinger for reading the manuscript; his inquiring about our algorithm in examples made us aware of an error in an earlier version of Sect. 12.

Before formulating our results in a general way (Sect. 1), it might be worth describing some of the ideas: Let X denote a hypersurface (defined locally by a single equation $f(x)=0$) in a manifold M. Let $a \in X$ and let $\mu_a(f)$ denote the order of vanishing of f at a; say $d=\mu_a(f)$. In this sketch, let us consider X to be *nonsingular* at a if $f = z^d$ (in germs at a; necessarily, $\mu_a(z)=1$). In general, we can choose coordinates $x=(x_1,\ldots,x_n)$ such that $\partial^d f/\partial x_n^d \neq 0$ in a neighbourhood of a; then the equation $(\partial^{d-1} f/\partial x_n^{d-1})(x) = 0$ defines a submanifold N of codimension 1 (in this neighbourhood); cf. Sect. 3. Let c_q be the restriction to N of $\partial^q f/\partial x_n^q$, $0 \le q \le d-2$. (For example, if $f(x) = c_0(\tilde{x}) + c_1(\tilde{x})x_n + \cdots + c_{d-1}(\tilde{x})x_n^{d-1} + x_n^d$, where $\tilde{x} = (x_1,\ldots,x_{n-1})$, and we assume by completing the d'th power that $c_{d-1} \equiv 0$, then N is given by $x_n = 0$ and the coefficients c_q have the meaning above.) The c_q are regular functions on N, as in (0.2).

Let $S_{(f,d)}$ denote the "equimultiple locus" $\{x : \ \mu_x(f) = d\}$. It is easy to see that $S_{(f,d)} = S_{\mathscr{H}}$, where $S_{\mathscr{H}} := \{x \in N : \ \mu_x c_q \ge d-q,\ q = 0,\ldots,d-2\}$ and $\mathscr{H} := \{(c_q, d-q)\}$. X is nonsingular at a if and only if all $c_q = 0$ (near a); i.e., $S_{(f,d)} = N$.

Consider the effect of a blowing-up σ with smooth centre $C \subset S_{(f,d)}$ (cf. Sect. 3). X lifts to a hypersurface X' (the "strict transform" of X), defined by $f'(y) := y_{\mathrm{exc}}^{-d} f\left(\sigma(y)\right)$, where y_{exc} denotes the "exceptional divisor" (i.e., the "exceptional hypersurface" $H = \sigma^{-1}(C)$ is given by $y_{\mathrm{exc}} = 0$). The corresponding transformation of coefficients is

$$c_q' = y_{\mathrm{exc}}^{-(d-q)} c_q \circ \sigma, \qquad 0 \le q \le d-2, \tag{0.3}$$

as functions (strictly speaking, defined locally up to an invertible factor), on the strict transform N' of N; clearly $\sigma(N') \subset N$. The formula for strict transform shows that if $a' \in \sigma^{-1}(a)$, then $\mu_{a'}(f') \leq d$, and if $\mu_{a'}(f') = d$ then $a' \in N'$ and $S_{(f',d)} = S_{\mathscr{H}'}$, where $\mathscr{H}' := \{(c'_q, d-q)\}$ (cf. 5.1 and 4.12). Likewise after a finite sequence of blowings-up with centres in the equimultiple loci of the successive strict transforms.

Suppose, for example, that

$$c_q(x)^{\frac{d!}{d-q}} = (x^{\Omega})^{d!} c_q^*(x), \qquad 0 \leq q \leq d-2, \tag{0.4}$$

where $\Omega = (\Omega_1, \ldots, \Omega_{n-1})$ and each $d!\Omega_i$ is a nonnegative integer, $x^{\Omega} := x_1^{\Omega_1} \cdots x_{n-1}^{\Omega_{n-1}}$ (with respect to coordinates $(x_1, \ldots, x_{n-1})$ of N) and some $c_q^*(a) \neq 0$. ((0.4) reflects the transformation law (0.3); $d!$ is used to factor with all c_q^* regular. We will write $d(a) = d$, $\Omega(a) = \Omega$.) Then $S_{(f,d)} = \{x \in N : \mu_x x^{\Omega} \geq 1\}$. So $S_{(f,d)} = \bigcup_I Z_I$, where $Z_I := \{x \in N : x_j = 0 \text{ if } j \in I\}$ and I runs over the *minimal* subsets of $\{1, \ldots, n-1\}$ such that $\sum_{j \in I} \Omega_j \geq 1$; i.e., over the subsets of $\{1, \ldots, n-1\}$ such that

$$0 \leq \sum_{j \in I} \Omega_j - 1 < \Omega_i, \qquad \text{for all } i \in I.$$

Suppose that σ is the blowing-up with centre $C = Z_I$, for some such I. If $a' \in \sigma^{-1}(a)$ and $\mu_{a'}(f') = d$, then $a' \in N'$; in this case, a' lies in a coordinate chart for N' in which σ has the following form, for some $i \in I$: $x_i = y_i$, $x_j = y_i y_j$ if $j \in I \setminus \{i\}$, and $x_j = y_j$ if $j \notin I$. Then each $(c'_q)^{d!/(d-q)} = (y^{\Omega'})^{d!} c_q^* \circ \sigma$, where $\Omega'_j = \Omega_j$ if $j \neq i$, and $\Omega'_i = \sum_{j \in I} \Omega_j - 1 < \Omega_i$. Since $1 \leq |\Omega'| < |\Omega|$, where $|\Omega| := \Omega_1 + \cdots + \Omega_{n-1}$, $d(a') < d(a)$ after at most $d!|\Omega(a)|$ such blowings-up (cf. proof of Theorem 1.14 in Sect. 6).

The question then is whether we can reduce to the hypothesis (0.4) by induction on dimension, replacing (f, d) in some sense by $\mathscr{H} = \{(c_q, d-q)\}$ on N. To set up the induction, we would have to consider from the start a collection $\mathscr{F} = \{(f, \mu_f)\}$ rather than a single pair (f, d). (A general X is, in any case, defined by several equations; cf. Ch. III.) Moreover, the transformation law (0.3) is not strict transform, so we would have to reformulate the original problem to not only desingularize f, but also make its "total transform" (its composite with the sequence of blowings-up) normal crossings (cf. (1.1) ff.). To this end, suppose that f actually represents the strict transform of our original function at some point in the history of the blowings-up involved, say where the order at a first becomes d. (We are following the transforms of the function at a sequence of points "a" over some original point.) Suppose there are $s = s(a)$ accumulated exceptional hypersurfaces H_p passing through a; say $H_p \cap N$ is defined on N by an equation $b_p(x) = 0$, $1 \leq p \leq s$. (Each $\mu_a(b_p) \geq 1$.) The transformation law for the b_p analogous to (0.3) is $b'_p = y_{\mathrm{exc}}^{-1} b_p \circ \sigma$. Suppose now that in (0.4) we also have $b_p(x)^{d!} = (x^{\Omega})^{d!} b_p^*(x)$, $p = 1, \ldots, s$ (and assume that either some $c_q^*(a) \neq 0$ or some $b_p^*(a) \neq 0$). Then the argument above shows that $\big(d(a'), s(a')\big) \leq \big(d(a), s(a)\big)$ (with respect to the lexicographic ordering of pairs), and that if $\big(d(a'), s(a')\big) = \big(d(a), s(a)\big)$ then $1 \leq |\Omega(a')| < |\Omega(a)|$. ($s(a')$ counts the exceptional divisors H_p' passing through a.)

The induction on dimension can be realized in various ways. In this article, we repeat our construction in increasing codimension to obtain $\mathrm{inv}_X(a) = \big(\nu_1(a), s_1(a); \nu_2(a), \ldots\big)$ such that $\big(\nu_1(a), s_1(a)\big) = \big(d(a), s(a)\big)$. (We recommend following the construction of inv_X on pp. 13-14 in parallel with Example 2.1.) $S_{\mathrm{inv}_X}(a) := \{x : \ \mathrm{inv}_X(x) = \mathrm{inv}_X(a)\}$ has the form $\{x \in N : \ \mu_x x^\Omega \geq 1\}$ where N is a submanifold (of codimension t, say), $x^\Omega = x_1^{\Omega_1} \cdots x_{n-t}^{\Omega_{n-t}}$ and each $\Omega_i \neq 0$ only if x_i is an exceptional divisor. Thus $S_{\mathrm{inv}_X}(a) = \bigcup Z_I$ where each Z_I is the intersection of $S_{\mathrm{inv}_X}(a)$ and all exceptional hypersurfaces containing Z_I. This local property implies each component of the maximum locus of inv_X is a global submanifold (Theorem 1.14, Remark 1.15. The invariant $\mu_X(a)$ of 1.14 corresponds to $|\Omega(a)|$ above.) Choosing a component of the maximum locus of inv_X as each successive centre of blowing up, we get the desingularization theorem 1.6 (and its generalizations in Ch. IV). In the language above: *Theorem. There is a mapping φ: $M' \to M$ realized as a composite of blowings-up with smooth centres such that φ is an isomorphism outside the singularities of X, the strict transform X' is smooth, and* $(\det d\varphi) \cdot (f \circ \varphi)$ *has only normal crossings. ($d\varphi$ is the Jacobian of φ.)*

The argument above, with far simpler versions of induction as in [BM3, Sect. 4] or [BM4], gives the same conclusion, but with φ a composite of mappings that are either blowings-up with smooth centres or surjections of the form $\coprod_j U_j \to \bigcup_j U_j$, where the latter is a locally finite open covering of a manifold and $\coprod$ means disjoint union.

1. An invariant for desingularization

Main results. Our invariant $\mathrm{inv}_X(a)$ is defined recursively over a sequence of blowings-up (or local blowings-up as in Sect. 4). Let X denote a space (as above) which is embedded in a manifold (smooth space) M. Consider a sequence of transformations

$$
\begin{array}{cccccccc}
\longrightarrow & M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow & \cdots \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 = M \\
 & X_{j+1} & & X_j & & & X_1 & & X_0 = X \\
 & E_{j+1} & & E_j & & & E_1 & & E_0 = \emptyset
\end{array}
\tag{1.1}
$$

where, for each j, σ_{j+1}: $M_{j+1} \to M_j$ denotes a blowing-up (or local blowing-up) with smooth centre $C_j \subset M_j$, X_{j+1} is the strict transform of X_j by σ_j (see Sect. 3) and E_{j+1} denotes the set of exceptional hypersurfaces. (E_{j+1} is the set of strict transforms of all $H \in E_j$, together with $\sigma_{j+1}^{-1}(C_j)$. When convenient, we will also use E_j to denote the union of the hypersurfaces H in E_j.) If $a \in M_j$, we set $E(a) = \{H \in E_j \ : \ a \in H\}$. (Throughout the article, by a point we mean a closed point (e.g., in the case of schemes). We adopt this convention for simplicity of exposition. Of course it suffices for the treatment of spaces in which closed points are dense; for example, schemes of finite type.)

Roughly speaking, the goal of "embedded resolution of singularities" is to find a finite sequence of blowings-up (1.1) (or a locally finite sequence in the case of noncompact analytic spaces) such that: If X' and E' denote the final strict transform of X and the final exceptional set (respectively), and if $\sigma\colon M' \to M$ denotes the composite of the sequence of blowings-up, then (1) X' is smooth; (2) $E' = \sigma^{-1}(\operatorname{Sing} X)$ (σ is an isomorphism outside E'); (3) X' and E' simultaneously have only normal crossings.

$\operatorname{Sing} X$ means the set of singular points of X. The condition (3) means that every point of M' admits a coordinate neighbourhood in which X' is a coordinate subspace and each hypersurface $H \in E'$ is a coordinate hypersurface.

Consider a tower of transformations (1.1). Our invariant $\operatorname{inv}_X(a)$, $a \in M_j$, $j = 0, 1, \ldots$, will be defined by induction on j, provided that the centres C_i, $i < j$, are *admissible* (or inv_X*-admissible*) in the sense that:

(1.2) (1) C_i and E_i simultaneously have only normal crossings;
(2) $\operatorname{inv}_X(\cdot)$ is locally constant on C_i.

The condition (1.2) (1) guarantees that E_{i+1} is a collection of smooth hypersurfaces having only normal crossings. The notation $\operatorname{inv}_X(a)$, where $a \in M_j$, indicates a dependence on the original space X and not merely on X_j. In fact, $\operatorname{inv}_X(a)$, $a \in M_j$, will be invariant under local isomorphisms of X_j which preserve $E(a)$ and certain subcollections $E^r(a)$. (We take the $E^r(a)$ to encode the history of the resolution process, as in "Presentation of the invariant" below in this section.) We can think of the desingularization algorithm as follows: $X \subset M$ determines $\operatorname{inv}_X(a)$, $a \in M$, and thus the first admissible centre of blowing-up $C = C_0$; then $\operatorname{inv}_X(a)$, $a \in M_1$, is defined and determines C_1, etc. The exceptional hypersurfaces serve as "global coordinate subspaces".

We can allow certain options in the definition of inv_X, but at this point we fix one definition in order to be concrete. $\operatorname{inv}_X(a)$, $a \in M_j$, will be a "word",

$$\operatorname{inv}_X(a) \;=\; \big(H_{X_j,a}, s_1(a);\, \nu_2(a), s_2(a);\, \ldots, s_t(a);\, \nu_{t+1}(a)\big)\ ,$$

beginning with the *Hilbert-Samuel function* $H_{X_j,a}$; i.e., the function

$$H_{X_j,a}(\ell) \;=\; \dim_{\underline{k}} \frac{\mathcal{O}_{X_j,a}}{\underline{m}^{\ell+1}_{X_j,a}}\ , \qquad \ell \in \mathbb{N}\,,$$

where $\underline{m}_{X_j,a}$ denotes the maximal ideal of $\mathcal{O}_{X_j,a}$. (In the case of schemes, we would replace $\dim_{\underline{k}}$ by length or $\dim_{\mathbb{F}_a}$ with respect to any embedding $\mathbb{F}_a \hookrightarrow \mathcal{O}_{X_j,a}/\underline{m}^{\ell+1}_{X_j,a}$, where $\mathbb{F}_a$ denotes the residue field $\mathcal{O}_{X_j,a}/\underline{m}_{X_j,a}$ of a.)

Remarks 1.3. $H_{X_j,a}(\ell)$, for ℓ large enough, coincides with a polynomial in ℓ of degree $\dim_a X_j$. (See Corollary 3.20.) $H_{X_j,a}(1) - H_{X_j,a}(0) = e_{X_j,a}$ is the minimal embedding dimension of X_j at a. (Thus $a \in \operatorname{Sing} X_j$ if and only if $e_{X_j,a} >$

$\dim_a X_j$.) If $a \notin \operatorname{Sing} X_j$, then $H_{X_j,a}(\ell) = \binom{e+\ell}{e}$ for all ℓ, where $e = e_{X_j,a} = \dim_a X_j$. If X_j is a hypersurface and $\dim_a M_j = n$, then

$$H_{X_j,a}(\ell) = \begin{cases} \binom{n+\ell}{n}, & \ell < \nu_{X_j,a}, \\ \binom{n+\ell}{n} - \binom{n+\ell-\nu_{X_j,a}}{n}, & \ell \geq \nu_{X_j,a}, \end{cases}$$

where $\nu_{X_j,a}$ is the *order* of X_j at a. ($\nu_{X_j,a} = \max\{\nu : \mathscr{I}_{X_j,a} \subset \underline{m}^{\nu}_{M_j,a}\}$, where $\mathscr{I}_{X_j,a}$ is the ideal of X_j at a.) In this case, we can therefore replace $H_{X_j,a}$ in the definition of $\operatorname{inv}_X(a)$ by $\nu_1(a) = \nu_{X_j,a}$.

The entries $s_r(a)$ of $\operatorname{inv}_X(a)$ are nonnegative integers reflecting the history of the accumulating exceptional hypersurfaces $\big(s_r(a) = \#E^r(a)\big)$, and the $\nu_r(a)$, $r \geq 2$, are "multiplicities" of "higher-order terms" of the equations of X_j at a; see "Presentation of the invariant" below. $\nu_2(a), \ldots, \nu_t(a)$ are quotients of positive integers whose denominators are bounded in terms of the previous part of $\operatorname{inv}_X(a)$. (More precisely, $e_{r-1}!\nu_r(a) \in \mathbb{N}$, $r = 2, \ldots, t$, where e_1 is the smallest integer k such that $H_{X_j,a}(\ell)$ coincides with a polynomial if $\ell \geq k$, and $e_r = \max\{e_{r-1}!,\ e_{r-1}!\nu_r(a)\}$.) The final entry $\nu_{t+1}(a) = 0$ or ∞, and $t \leq n = \dim_a M_j$. (The successive pairs $\big(\nu_r(a), s_r(a)\big)$ can be defined inductively using functions of $n - r + 1$ variables, so that $t \leq n$ by exhaustion of variables.) (In [BM3, Sect. 4] and [BM4, Sect. 3], the notation (d, r) is used for (ν_1, s_1).)

Example 1.4. Let $X \subset \underline{k}^n$ denote the hypersurface $x_1^{d_1} + x_2^{d_2} + \cdots + x_t^{d_t} = 0$, where $2 \leq d_1 \leq d_2 \leq \cdots \leq d_t$, $t \leq n$. Then

$$\operatorname{inv}_X(0) = \left(d_1, 0;\ \frac{d_2}{d_1}, 0;\ \cdots;\ \frac{d_t}{d_{t-1}}, 0;\ \infty\right).$$

(This is $\operatorname{inv}_X(0)$ at the origin 0 of $\underline{k}^n$ in "year zero"; i.e., before any blowings-up.)

Remark 1.5. To define inv_X, we need only work with a category $\mathscr{A}$ of local-ringed spaces $X = (|X|, \mathscr{O}_X)$ over $\underline{k}$ such that, for each $a \in |X|$: (1) The natural homomorphism $\mathscr{O}_{X,a} \to \widehat{\mathscr{O}}_{X,a}$ into the completion $\widehat{\mathscr{O}}_{X,a} := \varprojlim \mathscr{O}_{X,a}/\underline{m}^{k+1}_{X,a}$ is injective. (2) (The residue field $\mathbb{F}_a := \mathscr{O}_{X,a}/\underline{m}_{X,a}$ is included in $\widehat{\mathscr{O}}_{X,a}$) and $n = \dim_{\mathbb{F}_a} \underline{m}_{X,a}/\underline{m}^2_{X,a} < \infty$.

If $a \in |X|$, then $\operatorname{inv}_X(a)$ depends only on $\widehat{\mathscr{O}}_{X,a}$. More generally, $\operatorname{inv}_X(\cdot)$ is defined recursively over a sequence of formal local blowings-up, $\widehat{\mathscr{O}}_{X,a} = \widehat{\mathscr{O}}_{X_0,a_0} \xrightarrow{\sigma_1^*} \widehat{\mathscr{O}}_{X_1,a_1} \longrightarrow \cdots \xrightarrow{\sigma_j^*} \widehat{\mathscr{O}}_{X_j,a_j} \longrightarrow \cdots$, which are "admissible" ((1.2)). For each j, $\operatorname{inv}_X(a_j)$ depends only on $\widehat{\mathscr{O}}_{X_j,a_j}$ and the $E^r(a)$, as above. There is an ideal $\widehat{\mathscr{I}_{S_j}}$ of $\widehat{\mathscr{O}}_{X_j,a_j}$ corresponding to a formal "infinitesimal locus" $S_j = S_{\operatorname{inv}_X}(a_j)$ of points $x \in |X_j|$ such that $\operatorname{inv}_X(x) = \operatorname{inv}_X(a_j)$; S_j has only normal crossings. If we choose any component of S_j as the centre of σ^*_{j+1}, successively for

$j = 0, 1, \ldots,$ then we get an admissible sequence of formal local blowings-up leading to desingularization of $\widehat{\mathcal{O}}_{X,a}$. In order that the algorithm apply to $\mathcal{O}_{X,a}$, we need to impose conditions on our category $\mathscr{A}$ to guarantee that $\widehat{\mathcal{I}_{S_j}}$ is generated by an ideal $\mathcal{I}_{S_j} \subset \mathcal{O}_{X_j,a_j}$, and $S_j = V(\mathcal{I}_{S_j})$, where $V(\mathcal{I}_{S_j}) = \{x \in |X_j| : f(x) = 0,$ for all $f \in \mathcal{I}_{S_j}\}$ (as a germ at a).

To obtain global desingularization, we need to further restrict $\mathscr{A}$ so that inv_X, defined over an admissible sequence of blowings-up $\cdots \longrightarrow X_j \xrightarrow{\sigma_j} \cdots \longrightarrow X_1 \xrightarrow{\sigma_1} X_0$, takes only finitely many maximal values on each X_j (at least locally), and its maximum locus coincides in germs at any point a_j with $S_{\mathrm{inv}_X}(a_j)$ as above. See also (3.9).

The simplest form of our embedded desingularization theorem is the following:

Theorem 1.6 (cf. [H1, Main Theorem I]). *Suppose that $|X|$ is quasi-compact. Then there is a finite sequence of blowings-up (1.1) with smooth* inv_X*-admissible centers C_j such that:*

(1) For each j, either $C_j \subset \mathrm{Sing}\, X_j$ or X_j is smooth and $C_j \subset X_j \cap E_j$.

(2) Let X' and E' denote the final strict transform of X and exceptional set, respectively. Then X' is smooth and X', E' simultaneously have only normal crossings.

("Quasi-compact" means every open covering has a finite subcovering; "compact" means "Hausdorff and quasi-compact".) The conclusion of 1.6 holds, more generally, for $X|U$, where U is any relatively quasi-compact open subset of $|X|$. If X is a non-compact analytic space (for example, over $\mathbb{R}$ or $\mathbb{C}$; see also Theorem 13.3), Theorem 1.6 holds with a locally finite sequence of blowings-up. If σ is the composite of the sequence of blowings-up σ_j, then E' is the critical locus of σ, and $E' = \sigma^{-1}(\mathrm{Sing}\, X)$.

Remarks 1.7. (1) Our proof of Theorem 1.6 requires the hypotheses that, for X in our class of spaces, $\mathrm{Sing}\, X$ is closed and $H_{X,\cdot}$ is upper-semicontinuous, both with respect to the Zariski topology of $|X|$ (the topology whose closed sets are of the form $|Y|$, for any closed subspace Y of X; see Sect. 3.) We give a very simple proof of semicontinuity of $H_{X,\cdot}$ in Chapter III (Theorem 9.2; cf. [Ben]), for X in any of the classes of (0.1) (1), (2); in these classes, $\mathcal{O}_X$ is a coherent sheaf of rings, and it follows that $\mathrm{Sing}\, X$ is Zariski-closed (Proposition 10.1). Both hypotheses above are clear in the hypersurface case, for all classes of (0.1). (See [GD] for definitions of sheaf-theoretic terms like "coherent".)

(2) Theorem 1.6 resolves the singularities of X in a meaningful geometric sense provided $\mathrm{Reg}\, X := |X| \backslash \mathrm{Sing}\, X$ is Zariski-dense in $|X|$. We will say X is a *geometric space* if $\mathrm{Reg}\, X$ is Zariski-dense in $|X|$. For example, *reduced* complex analytic spaces or schemes of finite type are geometric. Suppose X is a geometric space. If $\sigma\colon M' \to M$ is a blowing-up with smooth centre C, we define the *geometric strict transform* X'' of X by σ as the smallest closed subspace Z of $\sigma^{-1}(X)$ such that $|Z| \supset |\sigma^{-1}(X)| \backslash |\sigma^{-1}(C)|$. (The strict and geometric strict

transforms coincide for reduced schemes or complex analytic spaces.) We can reformulate Theorem 1.6 to resolve the singularities of X by transformations which preserve the class of geometric spaces (Sect. 10): "geometric strict transform" can be used throughout the desingularization algorithm because, if X' denotes the strict transform of X, then $X'' \subset X'$ and, if $a' \in X''$, then $H_{X'',a'} \leq H_{X',a'}$ with equality if and only if $X'' = X'$ in germs at a' (cf. 1.14 below).

(3) In the categories of (0.1) (1) and (2), algebraic techniques make it possible to use our desingularization algorithm to prove theorems more precise than 1.6; for example, for spaces that are not necessarily reduced (Sect. 11). Theorem 1.6 does not exclude the possibility of blowing-up "resolved points"; i.e., a centre of blowing-up C_j may include points where X_j is smooth and has only normal crossings with respect to E_j. (See Example 2.3.) We can modify inv_X to avoid blowing up resolved points; see Sect. 12.

Our desingularization theorems are presented in Chapter IV. (To be brief, we concentrate in this introduction on an embedded space $X \hookrightarrow M$; see Theorem 13.2 for universal "embedded desingularization" of an abstract space X.) We give a constructive definition of inv_X in Chapter II. (The main idea is presented later in this introduction.)

Remark 1.8. *Transforming an ideal to normal crossings* (cf. [H1, Main Theorem II]): Suppose that $\mathcal{I} \subset \mathcal{O}_M$ is a sheaf of ideals of finite type. Let $\nu_1(a)$ denote the *order* $\nu_{\mathcal{I},a}$ of $\mathcal{I}$ at $a \in M$. ($\nu_{\mathcal{I},a} := \max\{\nu : \mathcal{I}_a \subset \underline{m}_{M,a}^{\nu}\}$.) If $\sigma : M' \to M$ is a local blowing-up with smooth centre C, we can define a *weak transform* $\mathcal{I}' \subset \mathcal{O}_{M'}$ of $\mathcal{I}$ by σ as follows: For all $a' \in M'$, $\mathcal{I}'_{a'}$ is the ideal generated by $y_{\mathrm{exc}}^{-\nu} f \circ \sigma$, $f \in \mathcal{I}_{\sigma(a')}$, where ν denotes the generic value of $\nu_1(a)$ on C (and y_{exc} is a local generator of the ideal of $\sigma^{-1}(C)$ at a'). In this context, our construction can be used to extend $\mathrm{inv}_{1/2}(\cdot) = \nu_1(\cdot)$ to an invariant $\mathrm{inv}_{\mathcal{I}}(\cdot)$ which is defined inductively over a sequence of transformations

$$
(1.9) \qquad \begin{array}{ccccccccc}
\longrightarrow & M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 = M \\
& E_{j+1} & & E_j & & & & E_1 & & E_0 = \emptyset \\
& \mathcal{I}_{j+1} & & \mathcal{I}_j & & & & \mathcal{I}_1 & & \mathcal{I}_0 = \mathcal{I}
\end{array}
$$

where the σ_{j+1} are local blowings-up whose centres are $\mathrm{inv}_{\mathcal{I}}$*-admissible* (cf. (1.2)), E_{j+1} is the set of exceptional hypersurfaces, and each $\mathcal{I}_{j+1}$ is the weak transform of $\mathcal{I}_j$. (See 1.18.) Using $\mathrm{inv}_{\mathcal{I}}$, our algorithm gives the following theorem (which is a consequence of Theorem 1.6 in the case that $\mathcal{I} = \mathcal{I}_X$ is the ideal sheaf of a hypersurface X).

Theorem 1.10. *Suppose that $|M|$ is quasi-compact. Then there is a finite sequence (1.9) of blowings-up σ_j, $j = 1, \ldots, k$, with smooth* $\mathrm{inv}_{\mathcal{I}}$*-admissible centres, such that $\mathcal{I}_k = \mathcal{O}_{M_k}$ and $\sigma^{-1}(\mathcal{I}) := \sigma^*(\mathcal{I}) \cdot \mathcal{O}_{M_k}$ is a normal-crossings divisor, where $\sigma : M_k \to M$ denotes the composite of the σ_j. ("Normal-crossings divisor" means a principal ideal of finite type, generated locally by a monomial in suitable coordinates.)*

It follows that if $\mathscr{J}_\sigma \subset \mathscr{O}_{M_k}$ denotes the ideal generated by the Jacobian determinant of σ, then $\mathscr{J}_\sigma \cdot \sigma^{-1}(\mathscr{T})$ is a normal-crossings divisor.

Remark 1.11. In year zero, there is a straightforward geometric definition of inv_X: Assume that X is a hypersurface. (The following construction will be extended to the general case in Remark 3.25, using the "diagram of initial exponents".) Locally, X is defined by a single equation $f = 0$. Consider the Taylor expansion $f(x) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha x^\alpha$ of f at a point a, *for a given coordinate system* $x = (x_1, \dots, x_n)$. (Say $x(a) = 0$. If $\alpha \in \mathbb{N}^n$, then x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. $\mathbb{N}$ denotes the nonnegative integers.) We associate to the Taylor expansion of f at a its *Newton diagram* or *support* $\mathfrak{Q}(a) = \{\alpha \in \mathbb{N}^n : f_\alpha \neq 0\}$. Let us order the hyperplanes H in $\mathbb{R}^n$ lexicographically with respect to $d = (d_1, \dots, d_n)$, where the d_i are the intersections of H with the coordinate axes, listed so that $d_1 \le d_2 \le \cdots \le d_n \le \infty$. We regard $\mathfrak{Q}(a)$ as a subset of the positive orthant of $\mathbb{R}^n$, and let $d(x) = (d_1, \dots, d_n)$ denote the maximum order of a hyperplane H which lies *under* $\mathfrak{Q}(a)$ (in the sense that for each $(\alpha_1, \dots, \alpha_n) \in \mathfrak{Q}(a)$, there exists $(\beta_1, \dots, \beta_n) \in H$ such that $\beta_i \le \alpha_i$ for each i); in particular, $0 < d_1 < \infty$. Of course, $d(x)$ depends on the coordinate system $x = (x_1, \dots, x_n)$. Set $d = \sup_{\substack{\text{coordinate} \\ \text{systems } x}} d(x)$, $d = (d_1, \dots, d_n)$. Then

$$\mathrm{inv}_X(0) = \left(d_1, 0; \frac{d_2}{d_1}, 0; \cdots; \frac{d_t}{d_{t-1}}, 0; \infty\right),$$

where d_t is the last finite d_i. It is natural to ask whether $d = \sup d(x)$ is realized by a particular coordinate system x. (In Example 1.4 above, the supremum is realized by the given coordinates.) The construction we use to define inv_X in Chapter II (or below in this section) gives a positive answer. Moreover, beginning with any coordinate system, we find an explicit change of variables to obtain coordinates $x = (x_1, \dots, x_n)$ in which $d(x) = d$; in these coordinates, the centre of the first blowing-up in our resolution algorithm is $x_1 = \cdots = x_t = 0$ (where the coordinates are indexed so that d_i corresponds to x_i, for each i). Consider another coordinate system $y = (y_1, \dots, y_n)$ in which the supremum d is realized (indexed again so that d_i corresponds to y_i, for each i). Write $y = \varphi(x)$, $\varphi = (\varphi_1, \dots, \varphi_n)$, for the coordinate transformation. Then, using $w_i = d_1/d_i$ as weights for the x_i and y_i, $i = 1, \dots, n$ (cf. Remark 3.25), the weighted initial forms of f with respect to x and y are obtained one from the other by the substitution $y = \varphi_w(x)$, where each $y_i = \varphi_{w,i}(x)$ is the weighted homogeneous part of order w_i in the Taylor expansion of $y_i = \varphi_i(x)$. (This remark will not be used here; we plan to pursue it elsewhere.)

1.12. A combinatorial analogue of resolution of singularities (cf. toroidal desingularization). Let $\mathscr{M}$ be a finite simplicial complex. We define the *blowing-up* σ_Σ of $\mathscr{M}$ *along* a simplex Σ as the smallest simplicial subdivision $\mathscr{M}'$ of $\mathscr{M}$ which includes the barycentre of Σ. If $V(\mathscr{M})$ denotes the set of vertices (0-simplices) $\{H_1, \dots, H_d\}$ of $\mathscr{M}$, then $V(\mathscr{M}') = \{H_1', \dots, H_{d+1}'\}$, where $H_k' = H_k$, $k \le d$, and H_{d+1}' is the barycentre of Σ. If D is a function D:

$V(\mathscr{M}) \to \mathbb{Z}$, we define the transform D' of D by σ_Σ as $D'\colon V(\mathscr{M}') \to \mathbb{Z}$, where $D'(H_k') = D(H_k)$, $k \le d$, and $D'(H_{d+1}') = \sum_{H_k \in V(\Sigma)} D(H_k)$. If $D_1, D_2\colon V(\mathscr{M}) \to \mathbb{Z}$, we say that $D_1 \le D_2$ if $D_1(H_k) \le D_2(H_k)$ for all k. *Theorem. Suppose $D_j\colon V(\mathscr{M}) \to \mathbb{Z}$, $j = 1, \dots, s$. Then there is a finite sequence of simplicial blowings-up of $\mathscr{M}$ after which the transforms D_j' of the D_j are locally totally ordered as follows: Let $\mathscr{M}'$ denote the final transform of $\mathscr{M}$. Then, for every simplex Σ of $\mathscr{M}'$, there is a permutation $(j_1, \dots, j_s)$ of the indices j such that $D_{j_1}'|V(\Sigma) \le D_{j_2}'|V(\Sigma) \le \cdots \le D_{j_s}'|V(\Sigma)$.*

A finite simplicial complex can be associated to a system of smooth hypersurfaces with only normal crossings in a smooth ambient space. Let M be a manifold and let E denote a finite collection of smooth hypersurfaces $\{H_1, \dots, H_d\}$ in M having only normal crossings. We associate to E the simplicial complex $\mathscr{M} = \mathscr{M}(E)$ whose vertices correspond to the H_k and whose simplices Σ correspond to nonempty intersections $H_{k_1} \cap \cdots \cap H_{k_q}$. Every finite simplicial complex can be realized in this way. Say that a blowing-up $\sigma\colon M' \to M$ is *admissible* if its centre C is an intersection of hypersurfaces in E; i.e., $C = C(\Sigma)$ corresponds to a simplex Σ of $\mathscr{M}$. The system of hypersurfaces E transforms under σ to $E' = \{H_1', \dots, H_{d+1}'\}$, where H_k' denotes the strict transform of H_k, $k \le d$, and $H_{d+1}' = \sigma^{-1}(C)$. It is easy to see that $\mathscr{M}' = \mathscr{M}(E')$ is the simplicial blowing-up σ_Σ of $\mathscr{M}$. A formal divisor $D = \sum_{k=1}^{d} n_k[H_k]$ (where each $n_k \in \mathbb{Z}$) on M corresponds to the function $D(H_k) = n_k$ on $V(\mathscr{M})$. The (*total*) *transform* of $D = \sum n_k[H_k]$ *by* σ is defined as $D' = \sum_{k=1}^{d} n_k[H_k'] + \big(\sum_{H_k \in V(\Sigma)} n_k\big)[H_{d+1}']$. Clearly, this is the same as the combinatorial transformation rule above. The preceding theorem is equivalent to the following "combinatorial desingularization theorem". (Compare to the role played by Lemma 4.7 in [BM3, Sect. 4]. See also Lemma 12.8.)

Theorem 1.13. *Let M be a manifold and $E = \{H_1, \dots, H_d\}$ a finite collection of smooth hypersurfaces in M having only normal crossings. Suppose we have a system of formal divisors $D_j = \sum n_{jk}[H_k]$ (where each $n_{jk} \in \mathbb{Z}$), $j = 1, \dots, s$. Then there is a finite sequence of admissible blowings-up of M after which the transforms D_j' of the D_j are locally totally ordered in the following sense: Let M', E' denote the final transforms of M, E. Then, for each $a' \in M'$, there is a permutation $(j_1, \dots, j_s)$ of the indices j such that $D_{j_1}'(H) \le D_{j_2}'(H) \le \cdots \le D_{j_s}'(H)$ for all $H \in E'$ with $a' \in H$.*

Proof. It is enough to consider $s = 2$. Our proof is a simple parallel of the construction in Chapter II (and Theorem 1.14). Let $a \in M$. Set $\nu_1(a) := \min\{\sum_{H \ni a} D_1(H), \sum_{H \ni a} D_2(H)\} - \sum_{H \ni a} \min\{D_1(H), D_2(H)\}$, and $\mathrm{inv}(a) := \nu_1(a)$. ("$s_1(a)$" is unneeded; "$\nu_2(a)$" = 0.) Put

$$\mu_2(a) := \max\Big\{\sum_{H \ni a} D_1(H),\ \sum_{H \ni a} D_2(H)\Big\} - \sum_{H \ni a} \min\{D_1(H), D_2(H)\}\ .$$

Let S_a denote (the germ at a of) $\{x \in M : \nu_1(x) = \nu_1(a)\}$. Then the irreducible components Z of S_a are of the form $Z = Z_I := \bigcap_{H \in I} H$ for certain $I \subset E$ (as germs at a) (and $Z = S_a \cap \bigcap_{H \supset Z} H$; cf. Theorem 1.14 (3)): To be explicit, say that $\nu_1(a) = \sum_{H \ni a} \big(D_1(H) - D_0(H)\big)$, where $D_0(H) := \min\{D_1(H), D_2(H)\}$. Set $J_k(a) := \{H : H \ni a \text{ and } D_k(H) > D_0(H)\}$, $k = 1, 2$. Then each $Z = Z_I$, where $I = J_1(a) \cup J$ and J is a subset of $J_2(a)$ that is minimal with respect to the property that $\sum_{H \in J} \big(D_2(H) - D_0(H)\big) \geq \nu_1(a)$. (In particular, if $\mu_2(a) = \nu_1(a)$, then $S_a = Z_I$, where $I = J_1(a) \cup J_2(a)$.)

Now let ν_1 be the maximum value of the invariant $\mathrm{inv}(a)$, $a \in M$, and let $S := \{x \in M : \nu_1(x) = \nu_1\}$. Then the irreducible components of S are the Z_I above, for all $a \in S$. Write $\mu_2(I) := \min_{a \in Z_I} \mu_2(a)$; then $\mu_2(I) = \max\{\sum_{H \in I} (D_1(H) - D_0(H)), \sum_{H \in I} (D_2(H) - D_0(H))\} \geq \nu_1$. Let σ be the blowing-up with centre one of these components Z_I. We claim that $\big(\nu_1(b); \mu_2(b)\big) < \big(\nu_1(\sigma(b)); \mu_2(\sigma(b))\big)$, for all $b \in \sigma^{-1}(Z_I)$ (so the theorem follows by induction). Indeed, by the minimality property above, $I = J_1 \cup J_2$, where $J_k := \{H \in I : D_k(H) > D_0(H)\}$, $k = 1, 2$. Say that $\nu_1 = \sum_{H \in I} \big(D_1(H) - D_0(H)\big)$. Let $a \in Z_I$. If $\nu_1 < \mu_2(I)$, then $J_1 = J_1(a)$; if $\nu_1 = \mu_2(I)$, then we can assume the same is true by interchanging $k = 1$ and 2 if necessary. In any case, $0 \leq \sum_{H \in I} \big(D_2(H) - D_0(H)\big) - \nu_1 < D_2(H_*) - D_0(H_*)$ for every $H_* \in J_2$. Let $b \in H_I := \sigma^{-1}(Z_I)$ and $a = \sigma(b)$. Then $D_1(H_I) := \sum_{H \in I} D_1(H) \leq \sum_{H \in I} D_2(H) =: D_2(H_I)$; hence $D_0(H_I) = D_1(H_I)$, and $D_2(H_I) - D_0(H_I) < D_2(H_*) - D_0(H_*)$ for every $H_* \in J_2$. If $H \in E$, let H' be the strict transform of H. Since $D_1(H_I) - D_0(H_I) = 0$, it follows that $\nu_1(b) \leq \nu_1(a)$ and $\nu_1(b) = \nu_1(a)$ if and only if $b \in \bigcap_{H \in J_1} H'$ and $\mu_2(b) = \sum_{H \ni b} \big(D_2(H) - D_0(H)\big) \geq \sum_{H \ni b} \big(D_1(H) - D_0(H)\big) = \nu_1(b)$ (in particular, $b \notin H'_*$, for some $H_* \in J_2$). But $\mu_2(a) = \sum_{H \ni a} \big(D_2(H) - D_0(H)\big)$ since $\nu_1 = \nu_1(a)$ and $J_1 = J_1(a)$. Therefore, $\nu_1(b) = \nu_1(a)$ implies that $\mu_2(b) \leq \mu_2(a) - \big(D_2(H_*) - D_0(H_*)\big) + \big(D_2(H_I) - D_0(H_I)\big) < \mu_2(a)$. (Since $\nu_1(b) \leq \mu_2(b)$, it follows that $\nu_1(b) < \nu_1(a)$ if $\mu_2(a) = \nu_1(a)$.) $\square$

Fundamental properties of inv_X. Let X denote a closed subspace of a smooth space M, as before. Our desingularization theorems will follow from four key properties satisfied by inv_X, for any admissible sequence of blowings-up (Theorem 1.14).

A function $\tau\colon |M| \to \Sigma$, where Σ is a partially-ordered set, will be called *Zariski-semicontinuous* if τ locally takes only finitely many values and, for all $\sigma \in \Sigma$, $S_\sigma := \{x \in |M| : \tau(x) \geq \sigma\}$ is Zariski-closed. (See Lemma 3.10, Definition 3.11.)

The function $\tau(\cdot) = H_{X,\cdot}$ takes values in the set $\mathbb{N}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to itself. $\mathbb{N}^{\mathbb{N}}$ is partially ordered as follows: If $H_1, H_2 \in \mathbb{N}^{\mathbb{N}}$, then $H_1 < H_2$ if $H_1(\ell) \le H_2(\ell)$ for all ℓ, and $H_1(\ell) < H_2(\ell)$ for some ℓ. We can then use the lexicographic ordering of words like $\mathrm{inv}_X(a)$ to obtain a partially-ordered set in which $\mathrm{inv}_X(\cdot)$ takes values.

Theorem 1.14. *Consider any* inv_X*-admissible sequence of local blowings-up (1.1). The following properties hold.*

(1) Semicontinuity. *(i) For each j, every point of $|M_j|$ admits a neighbourhood U such that* inv_X *takes only finitely many values in U and, for all $a \in U$, $\{x \in U : \mathrm{inv}_X(x) \le \mathrm{inv}_X(a)\}$ is Zariski-open in $\big|M_j|U\big|$. (ii)* inv_X *is "infinitesimally upper-semicontinuous" in the sense that* $\mathrm{inv}_X(a) \le \mathrm{inv}_X\big(\sigma_j(a)\big)$ *for all $a \in M_j$, $j \ge 1$.*

(2) Stabilization. *Given $a_j \in M_j$ such that $a_j = \sigma_{j+1}(a_{j+1})$, $j = 0, 1, 2, \ldots$, there exists j_0 such that* $\mathrm{inv}_X(a_j) = \mathrm{inv}_X(a_{j+1})$ *when $j \ge j_0$. (In fact, any nonincreasing sequence in the value set of* inv_X *stabilizes.)*

(3) Let $a \in M_j$ and let $S_X(a)$ denote the germ at a (with respect to the Zariski topology) of $S_{\mathrm{inv}_X(a)}$ (so that $\mathrm{inv}_X(\cdot) = \mathrm{inv}_X(a)$ *on $S_X(a)$). Then $S_X(a)$ and $E(a)$ simultaneously have only normal crossings (i.e., there are local coordinates in which each is a union of coordinate subspaces). If* $\mathrm{inv}_X(a) = (\ldots;\infty)$*, then $S_X(a)$ is smooth. If* $\mathrm{inv}_X(a) = (\ldots;0)$ *and Z denotes an irreducible component of $S_X(a)$, then*

$$Z = S_X(a) \cap \bigcap\{H \in E(a) : Z \subset H\} .$$

(4) Let $a \in M_j$. If $\mathrm{inv}_X(a) = (\ldots;\infty)$ *and σ is the local blowing-up of M_j with centre $S_X(a)$, then* $\mathrm{inv}_X(a') < \mathrm{inv}_X(a)$ *for all $a' \in \sigma^{-1}(a)$. Otherwise, there is an additional invariant $\mu_X(a) \ge 1$ such that, if Z is an irreducible component of $S_X(a)$ and σ is the local blowing-up with centre Z, then* $\big(\mathrm{inv}_X(a'), \mu_X(a')\big) < \big(\mathrm{inv}_X(a), \mu_X(a)\big)$ *for all $a' \in \sigma^{-1}(a)$. ($e_t!\mu_X(a) \in \mathbb{N}$, with e_t as defined following 1.3.)*

Theorem 1.14 will be proved in Chapter II in the case that X is a hypersurface, and completed in Chapter III in the general case. Condition (1) (i) implies inv_X is Zariski-semicontinuous if $|X|$ is quasi-compact or if X is an analytic space over a locally compact field $\underline{k}$ (Remark 6.14). Note that, because of the bounds on the denominators of the terms $\nu_r(a)$ in $\mathrm{inv}_X(a)$, the stabilization property (2) of Theorem 1.14 is an immediate consequence of the corresponding property of the Hilbert-Samuel function. An elementary proof of stability of the Hilbert-Samuel function can be found in [BM4, Theorem 5.2.1]. The present article is self-contained except for this result and some elementary properties of the diagram of initial exponents, for which we give references in Sect. 3.

Remark 1.15. Let $a \in M_j$ and $U = \{x \in |M_j| : \mathrm{inv}_X(x) \le \mathrm{inv}_X(a)\}$. Then each irreducible component of $S_X(a)$ extends to a smooth (Zariski-) closed subset of U. This is a consequence of 1.14 (3): If $a \in M_j$, we label every component Z of $S_X(a)$ as Z_I, where $I = \{H \in E(a) : Z \subset H\}$. Consider any total ordering on the collection of all subsets I of E_j. For each $a \in M_j$, put $J(a) = \max\{I :$

Z_I is a component of $S_X(a)\}$; set $\mathrm{inv}_X^e(a) = \big(\mathrm{inv}_X(a); J(a)\big)$. Clearly, $\mathrm{inv}_X^e(\cdot)$ satisfies 1.14 (1)(i) and its maximum locus in U is smooth.

Of course, given $a \in M_j$ and any component Z_I of $S_X(a)$, we can choose the ordering above so that $I = J(a) = \max\{J : J \subset E_j\}$; therefore, Z_I extends to a smooth Zariski-closed subset of U.

Remark 1.16. The preceding construction shows that $\mathrm{inv}_X(a)$ can be extended to an invariant $\mathrm{inv}_X^e(a) = \big(\mathrm{inv}_X(a); J(a)\big)$ which has the property that, for all a, $S_X^e(a)$ is smooth (where $S_X^e(a)$ is the germ of $S_{\mathrm{inv}_X^e(a)}$ at a): It suffices to order the subsets of each E_j as follows: Write $E_j = \{H_1^j, \ldots, H_j^j\}$, where H_i^j is the strict transform of H_i^{j-1} by σ_j, $i = 1, \ldots, j-1$, and $H_j^j = \sigma_j^{-1}(C_{j-1})$ (i.e., each H_i^j is the strict transform of $\sigma_i^{-1}(C_{i-1})$ by the sequence of blowings-up $\sigma_{i+1}, \ldots, \sigma_j$). Associate to each $I \subset E_j$ the sequence $(\delta_1, \ldots, \delta_j)$, where $\delta_i = 0$ if $H_i^j \notin I$ and $\delta_i = 1$ if $H_i^j \in I$, and use the lexicographic ordering of such sequences, for all j and $I \subset E_j$. (See Remark 6.17.)

Universal and canonical desingularization. The extended invariant inv_X^e and Theorem 1.14 give a desingularization algorithm with uniquely determined centres of blowing up: When our spaces are quasi-compact (e.g., schemes or compact analytic spaces) we get a tower of inv_X-admissible blowings-up (1.1) by successively choosing as each smooth closed centre C_j, the locus of (the finitely many) maximal values of inv_X^e on $\mathrm{Sing}\, X_j$. (If $a \in \mathrm{Sing}\, X_j$, then $S_X(a) \subset \mathrm{Sing}\, X_j$ because the Hilbert-Samuel function distinguishes between smooth and singular points.) By property 1.14 (4), $\big(\mathrm{inv}_X(a'),\ \mu_X(a')\big) < \big(\mathrm{inv}_X(a),\ \mu_X(a)\big)$ for all $a \in C_j$ and $a' \in \sigma_{j+1}^{-1}(a)$. Theorem 1.6 follows. (See Sect. 10.) A theorem for analytic spaces X which are not necessarily compact follows from the algorithm applied to relatively compact open subsets of X (Sect. 13).

Our desingularization algorithm is *universal* for Noetherian spaces: To every X, we associate a morphism of resolution of singularities $\sigma_X \colon X' \to X$ such that any local isomorphism $X \mid U \to Y \mid V$ lifts to an isomorphism $X' \mid \sigma_X^{-1}(U) \to Y' \mid \sigma_Y^{-1}(V)$ (in fact, lifts to isomorphisms throughout the entire towers of blowings-up). (U, V denote Zariski-open subsets of $|X|$, $|Y|$, respectively.) See Sect. 13.

For analytic spaces which are not necessarily compact, the resulting procedure is *canonical*: Given X, there is a morphism of desingularization $\sigma_X \colon X' \to X$ such that any isomorphism $X|U \to X|V$ (over subsets U, V of $|X|$ which are open in the Hausdorff topology) lifts to an isomorphism $X' \mid \sigma_X^{-1}(U) \to X' \mid \sigma_X^{-1}(V)$. (See Sect. 13.)

Presentation of the invariant. We outline here the construction of inv_X that is detailed in Chapter II. (It might help to read this subsection in parallel with the examples of Sect. 2.) The entries $s_1(a), \nu_2(a), s_2(a), \ldots$ of $\mathrm{inv}_X(a)$ will themselves be defined recursively. Let us write inv_r for inv_X truncated after s_r (with the convention that $\mathrm{inv}_r(a) = \mathrm{inv}_X(a)$ if $r > t$). We also write $\mathrm{inv}_{r+\frac{1}{2}}(a) = (\mathrm{inv}_r; \nu_{r+1})$, so that $\mathrm{inv}_{1/2}(a)$ means $H_{X_j,a}$. For each r, the entries s_r, ν_{r+1} of inv_X can be defined inductively over a tower of (local) blowings-up (1.1) whose centres C_i are $(r - \frac{1}{2})$-*admissible* in the sense that:

(1.17)(1) C_i and E_i simultaneously have only normal crossings;
(2) $\mathrm{inv}_{r-\frac{1}{2}}$ is locally constant on C_i.

Once $\mathrm{inv}_{r+\frac{1}{2}}$ is defined, $r \geq 0$, s_{r+1} can be introduced immediately, in an invariant way: Consider a tower of local blowings-up (1.1) with $(r + \frac{1}{2})$-admissible centres. Write $\pi_{ij} = \sigma_{i+1} \circ \cdots \circ \sigma_j$, $i = 0, \ldots, j-1$, and $\pi_{jj} = id$. Suppose $a \in M_j$. We set $a_i = \pi_{ij}(a)$. First consider $r = 0$. Let i denote the smallest index k such that $\mathrm{inv}_{1/2}(a) = \mathrm{inv}_{1/2}(a_k)$ and set $E^1(a) = \{H \in E(a) : H$ is the strict transform of some hypersurface in $E(a_i)\}$. We define $s_1(a) = \#E^1(a)$. In general, suppose that i is the smallest index k such that $\mathrm{inv}_{r+\frac{1}{2}}(a) = \mathrm{inv}_{r+\frac{1}{2}}(a_k)$. Let $E^{r+1}(a) = \{H \in E(a) \setminus \bigcup_{q \leq r} E^q(a)$: H is the strict transform of some element of $E(a_i)\}$. We define $s_{r+1}(a) = \#E^{r+1}(a)$.

We will introduce each $\nu_{r+1}(a)$ by an explicit construction in local coordinates. Let us consider data of the following type at a (closed) point $a \in M$:

$N = N(a)$: a germ at a of a regular submanifold of M of codimension p;

$\mathcal{H}(a) = \{(h, \mu_h)\}$: a finite collection of pairs (h, μ_h), where each $h \in \mathcal{O}_{N,a}$ and each $\mu_h \in \mathbb{Q}$ is an "assigned multiplicity" $\mu_h \leq \mu_a(h)$. ($\mu_a(h)$ is the order of h at a);

$\mathcal{E}(a)$: a collection of smooth hypersurfaces $H \ni a$ such that N and $\mathcal{E}(a)$ simultaneously have only normal crossings, and $N \not\subset H$, for all $H \in \mathcal{E}(a)$.

We will call $\big(N(a), \mathcal{H}(a), \mathcal{E}(a)\big)$ an *infinitesimal presentation*, and we define its *equimultiple locus* $S_{\mathcal{H}(a)}$ as $\{x \in N : \mu_x(h) \geq \mu_h,$ for all $(h, \mu_h) \in \mathcal{H}(a)\}$. $S_{\mathcal{H}(a)} \subset N$ is well-defined as a germ at a. Given an infinitesimal presentation $\big(N(a), \mathcal{H}(a), \mathcal{E}(a)\big)$, we also define a transform $\big(N(a'), \mathcal{H}(a'), \mathcal{E}(a')\big)$ by a morphism of each of 3 types: (i) *admissible blowing-up*, (ii) *projection from the product with a line*, (iii) *exceptional blowing-up*. See (4.3). For example, a local blowing-up σ: $M' \to M$ with smooth centre C is admissible if $C \subset S_{\mathcal{H}(a)}$ and C and $\mathcal{E}(a)$ simultaneously have only normal crossings. In this case, let N' denote the strict transform of N by σ, and let $a' \in \sigma^{-1}(a)$ such that $a' \in N'$ and $\mu_{a'}(h') \geq \mu_h$, for all $(h, \mu_h) \in \mathcal{H}(a)$, where $h' = y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma$ (provided such a' exists). (y_{exc} denotes a local generator of the ideal of $\sigma^{-1}(C)$.) We set $N(a') =$ germ of N' at a', $\mathcal{H}(a') = \{(h', \mu_h)\}$, and $\mathcal{E}(a') = \{\sigma^{-1}(C)\} \cup \{H' : H \in \mathcal{E}(a),\ a' \in H'\}$ (where H' = the strict transform of H).

Transformations of types (ii) and (iii) will be needed only to prove the invariance of $\nu_{r+1}(a)$ using certain sequences of test blowings-up. Of course $y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma$ above is defined only up to an invertible factor, but two different choices are equivalent in the sense of the following definition: Given $\mathcal{E}(a)$, we will say that two infinitesimal presentations $\big(N, \mathcal{F}(a), \mathcal{E}(a)\big)$ and $\big(P, \mathcal{H}(a), \mathcal{E}(a)\big)$ (perhaps of different codimension) are *equivalent* (*with respect to transformations of* types (i), (ii) *and* (iii)) if:

(1) $S_{\mathcal{F}(a)} = S_{\mathcal{H}(a)}$.

(2) If σ is a local blowing-up as in (i) and $a' \in \sigma^{-1}(a)$, then $a' \in N'$ and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_f} f \circ \sigma) \geq \mu_f$, for all $(f, \mu_f) \in \mathscr{F}(a)$, if and only if $a' \in P'$ and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma) \geq \mu_h$, for all $(h, \mu_h) \in \mathscr{H}(a)$.

(3) After a transformation of type (i), (ii) or (iii), $\left(N', \mathscr{F}(a'), \mathscr{E}(a')\right)$ is equivalent to $\left(P', \mathscr{H}(a'), \mathscr{E}(a')\right)$. (This makes sense recursively.)

For example, assume that $(N(a), \mathscr{H}(a), \mathscr{E}(a))$ is an infinitesimal presentation, $\mathscr{H}(a) = \{(h, \mu_h)\}$. Then: (1) There is an equivalent presentation with $\mu_h \in \mathbb{N}$, independent of h: simply replace each (h, μ_h) by $(h^{e/\mu_h}, e)$, for suitable e. (2) Suppose there is $(h, \mu_h) \in \mathscr{H}(a)$ with $\mu_a(h) = \mu_h$ and $h = \Pi h_i^{m_i}$. If we replace (h, μ_h) in $\mathscr{H}(a)$ by the collection of (h_i, μ_{h_i}), each $\mu_{h_i} = \mu_a(h_i)$, then we obtain an equivalent presentation.

We will prove that

$$\mu_{\mathscr{H}(a)} := \min_{(h,\mu_h) \in \mathscr{H}(a)} \frac{\mu_a(h)}{\mu_h}$$

is an invariant of the equivalence class of the infinitesimal presentation $\left(N(a), \mathscr{H}(a), \mathscr{E}(a)\right)$ (in fact, with respect to transformations of types (i), (ii) alone).

Our construction starts with a local invariant that admits a presentation; we consider here the Hilbert-Samuel function $H_{X,\cdot}$ of our space $X \subset M$ (but see also 1.8, 1.18): We first introduce the transform X' of X by a morphism σ of type (i), (ii), (iii): X' is the strict transform of X in the case of (i), and the total transform $\sigma^{-1}(X)$ in the case of (ii) or (iii). An infinitesimal presentation $N = \left(N(a), \mathscr{H}(a), \mathscr{E}(a)\right)$ with $\operatorname{codim} N = p$ will be called a (*codimension* p) *presentation of* $H_{X,\cdot}$ *at* a (*with respect to* $\mathscr{E}(a)$) if:

(1) $S_{\mathscr{H}(a)} = S_H(a)$, where $S_H(a)$ denotes the germ at a of $\{x : H_{X,x} = H_{X,a}\}$.

(2) After an admissible local blowing-up σ ((i) above), $H_{X',a'} = H_{X,a}$ if and only if $a' \in N'$ and $\mu_{a'}(h') \geq \mu_h$ for all $(h, \mu_h) \in \mathscr{H}(a)$).

(3) Conditions (1) and (2) continue to hold after any sequence of transformations of types (i), (ii) and (iii).

In particular, after any sequence of transformations (i), (ii), (iii), the transform $(N(a'), \mathscr{H}(a'), \mathscr{E}(a'))$ is a (codimension p) presentation of $H_{X',\cdot}$ at a', with respect to $\mathscr{E}(a')$. Of course, any two presentations of $H_{X,\cdot}$ at a with respect to $\mathscr{E}(a)$ are equivalent. It is clear that the equivalence class of a presentation of $H_{X,\cdot}$ at a with respect to $\mathscr{E}(a)$ depends only on the local isomorphism class of M, X, $\mathscr{E}(a)$.

Consider, for example, a hypersurface $X \subset M$. Let $\mathrm{inv}_{1/2}(a) = \nu_1(a)$ be the order $\nu_{X,a}$ of X at a point a. Suppose that $g(x) = 0$ is a local equation of X at a (i.e., g generates $\mathscr{I}_{X,a}$). Let $N(a)$ = germ of M at a, $\mathscr{G}(a) = \{(g, d)\}$, where $d = \nu_1(a)$, and $\mathscr{E}(a) = \emptyset$. Then $\left(N(a), \mathscr{G}(a) = \mathscr{G}_1(a), \mathscr{E}(a)\right)$ is a codimension zero presentation of ν_1 at a. We define ν_2 (and the successive ν_{r+1}) by induction on codimension; the key point is that we can choose $z \in \mathscr{O}_{M,a}$ such that $\mu_a(z) = 1$ and $(N(a), \mathscr{G}(a), \mathscr{E}(a))$ is equivalent to $(N(a), \mathscr{G}(a) \cup \{(z, 1)\}, \mathscr{E}(a))$. It follows that, after any sequence of transformations of types (i), (ii) and (iii), $\mu_{a'}(z') = 1$, $S_{\mathscr{G}(a')} \subset \{z' = 0\}$ and $(N(a'), \mathscr{G}(a'), \mathscr{E}(a'))$ is equivalent to $(N(a'), \mathscr{G}(a') \cup \{(z', 1)\}, \mathscr{E}(a'))$ (Proposition 4.12).

To construct the element z above: Let $(x_1,\ldots,x_n)$ be a local coordinate system for M at a (cf. Sect. 3). (By a linear coordinate change) we can assume $(\partial^d g/\partial x_n^d)(a) \neq 0$. Take $z = \partial^{d-1}g/\partial x_n^{d-1}$; $z = 0$ defines a (germ of) a regular submanifold $N_1 = N_1(a)$ of M of codimension 1. If $\mathscr{C}_1(a)$ denotes the collection of pairs $(h,\mu_h) = \left(\frac{\partial^q g}{\partial x_n^q}\Big|N_1,\, d-q\right)$, $q = 0,\ldots,d-2$ (each h makes sense as an element of $\mathscr{O}_{N_1,a}$), then $\left(N_1(a),\mathscr{C}_1(a),\mathscr{E}(a)=\emptyset\right)$ is a codimension 1 presentation of ν_1 at a (cf. 4.18).

Now consider a sequence (1.1) with $\frac{1}{2}$-admissible centres C_j. Let $a \in M_j$. Let i be the smallest k such that $\nu_1(a) = \nu_1(a_k)$; in particular, $E(a_i) = E^1(a_i)$. Let $(N(a_i),\mathscr{G}_1(a_i),\mathscr{E}(a_i)=\emptyset)$ be a codimension zero presentation of ν_1 at a_i, and let $(N(a),\mathscr{G}_1(a),\mathscr{E}(a))$ be its transform at a (by the sequence of blowings-up $\sigma_{i+1},\ldots,\sigma_j$). Then $\mathscr{E}(a) = E(a)\backslash E^1(a)$ ($=\mathscr{E}_1(a)$, say), and $(N(a),\mathscr{G}_1(a),\mathscr{E}_1(a))$ is a codimension zero presentation of ν_1 at a with respect to $\mathscr{E}_1(a)$. For each $H \in E^1(a)$, let $\ell_H \in \mathscr{O}_{M_j,a}$ generate $\mathscr{I}_{H,a}$, and let $\mathscr{F}_1(a)$ denote $\mathscr{G}_1(a)$ together with all pairs $(f,\mu_f) = (\ell_H,1)$, $H \in E^1(a)$. Then $(N(a),\mathscr{F}_1(a),\mathscr{E}_1(a))$ is a codimension zero presentation of $\mathrm{inv}_1 = (\nu_1,s_1)$ at a.

As above, choose $z \in \mathscr{O}_{M_i,a_i}$ such that $\mu_{a_i}(z) = 1$ and $(N(a_i),\mathscr{G}_1(a_i),\mathscr{E}_1(a_i))$ is equivalent to $(N(a_i),\mathscr{G}_1(a_i)\cup\{(z,1)\},\mathscr{E}_1(a_i))$. If z' is the transform of z at a, then $(N(a),\mathscr{G}_1(a),\mathscr{E}_1(a))$ is equivalent to $(N(a),\mathscr{G}_1(a)\cup\{(z',1)\},\mathscr{E}_1(a))$, and therefore $(N(a),\mathscr{F}_1(a),\mathscr{E}_1(a))$ is equivalent to $(N(a),\mathscr{F}_1(a)\cup\{(z',1)\},\mathscr{E}_1(a))$. Suppose that $(x_1,\ldots,x_n)$ is a local coordinate system for M_j at a such that $(\partial z'/\partial x_n)(a) \neq 0$. Let $N_1 = N_1(a)$ denote the (germ at a of a) regular submanifold $\{z' = 0\}$, and $\mathscr{H}_1(a)$ the collection of pairs $(h,\mu_h) = \left(\frac{\partial^q f}{\partial x_n^q}|N_1,\mu_f - q\right)$, $0 \le q < \mu_f$, for all $(f,\mu_f) \in \mathscr{F}_1(a)$. Then $(N_1(a),\mathscr{H}_1(a),\mathscr{E}_1(a))$ is a codimension 1 presentation of inv_1 at a. (Likewise, if $\mathscr{C}_1(a)$ denotes the collection of pairs $\left(\frac{\partial^q g}{\partial x_n^q}|N_1,\mu_g - q\right)$, $0 \le q < \mu_g$, for all $(g,\mu_g) \in \mathscr{G}_1(a)$, then $(N_1(a),\mathscr{C}_1(a),\mathscr{E}_1(a))$ is a codimension 1 presentation of ν_1 at a.)

Suppose that $\left(N_1(a),\mathscr{H}_1(a),\mathscr{E}_1(a)\right)$ is any codimension 1 presentation of inv_1 at a, with respect to $\mathscr{E}_1(a) = E(a)\backslash E^1(a)$. Let $\mu_2(a) = \mu_{\mathscr{H}_1(a)}$. If $\mu_2(a) = \infty$, we set $\mathrm{inv}_X(a) = \left(\mathrm{inv}_1(a);\infty\right)$. Otherwise, for all $H \in \mathscr{E}_1(a)$, we write

$$\mu_{2H}(a) := \min\left\{\frac{\mu_{H,a}(h)}{\mu_h} :\ (h,\mu_h) \in \mathscr{H}_1(a)\right\}\ ,$$

where $\mu_{H,a}(h)$ denotes the *order of* h *along* $H \cap N_1$ *at* a (i.e., the order to which a generator x_H of the local ideal of $H \cap N_1$ factors from h); we define $\nu_2(a)$ as

$$\nu_2(a) := \mu_2(a) - \sum_H \mu_{2H}(a)\ .$$

Then $\nu_2(a) \ge 0$. We will prove that each $\mu_{2H}(a)$ and thus $\nu_2(a)$ is an invariant of the equivalence class of $\left(N_1(a),\mathscr{H}_1(a),\mathscr{E}_1(a)\right)$ (with respect to transformations (i), (ii) and (iii), but with a certain restriction on the sequence allowed; see 4.10);

hence each $\mu_{2H}(a)$ and $\nu_2(a)$ are invariants of the local isomorphism class of M_j, X_j, $E(a_j)$, $E^1(a_j)$.

Let $D(a) = \prod_{H\in\mathscr{E}_1(a)} x_H^{\mu_{2H}(a)}$, $D(a) = D_2(a)$; each $h \in \mathscr{H}_1(a)$ can be factored as $h = D^{\mu_h}\cdot g$, and $\mu_a(g) \geq \mu_g$, where $\mu_g = \mu_h \cdot \nu_2(a)$. (Rational exponents and orders can be avoided by raising to suitable powers.) Let $\mathscr{G}_2(a)$ be the collection of pairs $\{(g,\mu_g)\}$ together with $\big(D, 1-\nu_2(a)\big)$ if $\nu_2(a) < 1$. ($\mathscr{G}_2(a) := \{(D,1)\}$ in the case that $\nu_2(a) = 0$.) Then $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_1(a)\big)$ is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a with respect to $\mathscr{E}_1(a) = E(a)\backslash E^1(a)$. If $\nu_2(a) = 0$, set $\mathrm{inv}_X(a) = \mathrm{inv}_{1\frac{1}{2}}(a)$.

Suppose that $0 < \nu_2(a) < \infty$. Clearly, $\mu_{\mathscr{G}_2(a)} = 1$. Now assume that the σ_{j+1} in (1.1) are $1\frac{1}{2}$-admissible. Set $\mathscr{E}_2(a) = \mathscr{E}_1(a)\backslash E^2(a)$. Then $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_2(a)\big)$ is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a with respect to $\mathscr{E}_2(a)$ and, as above, there is an equivalent codimension 2 presentation $\big(N_2(a), \mathscr{C}_2(a), \mathscr{E}_2(a)\big), \ldots$. The construction can be repeated in increasing codimension. Eventually we reach $t \leq n = \dim_a M_j$ such that $0 < \nu_r(a) < \infty$ if $r \leq t$, and $\nu_{t+1}(a) = 0$ or ∞. Then we define $\mathrm{inv}_X(a) = (\mathrm{inv}_t(a); \nu_{t+1}(a))$ and $\mu_X(a) = \mu_{t+1}(a)$. See Chapter II. Our presentations satisfy a natural property of "semicoherence" (6.4) which allows us to prove that inv_X is Zariski-semicontinuous using the (elementary) Zariski-semicontinuity of order of a regular function. In Chapter II, we thus prove Theorem 1.14 in the case of a hypersurface.

Remark 1.18. In the context of Remark 1.8, we can obtain a codimension zero presentation $(N(a), \mathscr{G}(a), \mathscr{E}(a) = \emptyset)$ of $\nu_1 = \nu_{\mathscr{I}}$ at a (with respect to the notion of weak transform) simply by taking $N(a) =$ the germ of M at a, and $\mathscr{G}(a) = \{(g, \nu_1(a))\}$, where $\{g\}$ is any finite set of generators of $\mathscr{I}_a$. The construction above allows us to define $\mathrm{inv}_{\mathscr{I}}(\cdot)$ and thus to prove the analogue of Theorem 1.14, and Theorem 1.10.

Presentation of the Hilbert-Samuel function. In higher codimension, we can define inv_X exactly as in the case of a hypersurface, provided we find a (semi-coherent) presentation of the Hilbert-Samuel function. This is the subject of Chapter III. The standard basis of $\widehat{\mathscr{I}}_{X,a} \subset \widehat{\mathscr{O}}_{M,a}$ (with respect to any identification $\widehat{\mathscr{O}}_{M,a} \cong \underline{k}[[X_1,\ldots,X_n]]$) provides a *formal* presentation of $H_{X,\cdot}$ at a. The Henselian division theorem of Hironaka [H3] gives a presentation (at least with respect to admissible blowings-up (i); cf. [H3, Sect. 7, Theorem 1], [BM4, Theorem 7.3]) that is algebraic in the sense of Artin, and hence involves passing to an étale covering of X. We use an elementary division algorithm to get a presentation by regular functions. We also give $S_{H_{X,a}} = \{x \in |X| : H_{X,x} \geq H_{X,a}\}$ a natural structure of a closed *subspace* of X (cf. [Gi]), and prove equality of the ideals defining $S_{H_{X,a}}$ and the equimultiple locus of a regular presentation.

Remark 1.19. The standard basis of $\widehat{\mathscr{I}}_{X,a}$ itself extends to a presentation of the Hilbert-Samuel function which is regular in a weaker sense that nevertheless suffices to prove desingularization using Chapter II: Let $\{F\} \subset \widehat{\mathscr{I}}_{X,a}$ denote the standard basis (with respect to a generic coordinate system). Then all formal derivatives $\partial^{|\alpha|}F/\partial X^\alpha$, $\alpha \in \mathbb{N}^n$, when restricted to $S_{H_{X,a}}$, are (induced by) regular functions defined in a common neighbourhood U of a. Moreover, the induced formal expansions at each $b \in S_{H_{X,a}} \cap U$ provide a formal presentation at b. (This was our original approach and seems of independent interest; we plan to publish details elsewhere.)

2. Examples and an application

In the examples below, we will follow the desingularization algorithm (over a sequence of local blowings-up of a hypersurface) as sketched in "Presentation of the invariant" in Sect. 1 and detailed in Chapter II. We will use the notation from Sect. 1.

Example 2.1. Consider the hypersurface $X = V(g) \subset \underline{k}^3$ defined by $g(x) = x_3^2 - x_1^2x_2^3$.

Year zero. Let $a = 0$. Then $\nu_1(a) = \mu_a(g) = 2$ and $E(a) = \emptyset$, so $s_1(a) = 0$. A codimension zero presentation of $\mathrm{inv}_{1/2} = \nu_1$ at a (with respect to $\mathscr{E}_1(a) = \emptyset$) is given by $(N(a), \mathscr{G}_1(a), \mathscr{E}_1(a) = \emptyset)$, where $N(a) = \underline{k}^3$ and $\mathscr{G}_1(a) = \{(g, 2)\} = \mathscr{F}_1(a)$. We can take $N_1(a) = \{x_3 = 0\}$ and $\mathscr{H}_1(a) = \{(x_1^2x_2^3, 2)\}$ to get a codimension 1 presentation $(N_1(a), \mathscr{H}_1(a), \mathscr{E}_1(a))$ of $\mathrm{inv}_1 = (\nu_1, s_1)$ at a. Thus, $\nu_2(a) = \mu_2(a) = 5/2$ and $\mathrm{inv}_{1\frac{1}{2}}(a) = (2, 0; 5/2)$. Let $\mathscr{G}_2(a) = \{(x_1^2x_2^3, 5)\}$; then $(N_1(a), \mathscr{G}_2(a), \mathscr{E}_1(a))$ is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a. The latter is equivalent to $(N_1(a), \{(x_1, 1), (x_2, 1)\}, \mathscr{E}_1(a) = \emptyset)$, so repeating the construction, we find $\mathrm{inv}_X(a) = (2, 0; 5/2, 0; 1, 0; \infty)$ and $S_{\mathrm{inv}_X}(a) = S_{\mathrm{inv}_{1\frac{1}{2}}}(a) = \{a\}$. ($S_{\mathrm{inv}_X}(a)$ is the germ of $S_{\mathrm{inv}_X(a)}$ at a, etc.) We thus let $\sigma_1 : M_1 \to M_0 = \underline{k}^3$ be the blowing-up with centre $C_0 = \{a\}$. M_1 is covered by coordinate charts $U_i = M_1 \backslash \{x_i = 0\}'$, where $\{x_i = 0\}'$ is the strict transform of $\{x_i = 0\}$, $i = 1, 2, 3$; $\sigma_1|U_1$ can be written $x_1 = y_1$, $x_2 = y_1y_2$, $x_3 = y_1y_3$ (cf. "Blowing up" in Sect. 3).

Year one. Let X_1 denote the strict transform of X by σ_1; then $X_1 \cap U_1 = V(g_1)$, where $g_1 = y_1^{-2}g\circ\sigma_1 = y_3^2 - y_1^3y_2^3$. Let $b = 0$ in U_1. Then $\nu_1(b) = 2 = \nu_1(a)$; therefore, $E^1(b) = \emptyset$, $s_1(b) = 0$, and $\mathscr{E}_1(b) := E(b)\backslash E^1(b) = E(b) = \{H_1\}$, where $H_1 = \sigma_1^{-1}(a) = \{y_1 = 0\}$. We can take $\mathscr{F}_1(b) = \mathscr{G}_1(b) = \{(g_1, 2)\}$, $N_1(b) = \{y_3 = 0\} = N_1(a)'$, and $\mathscr{H}_1(b) = \{(y_1^3y_2^3, 2)\}$. (Of course $y_1^3y_2^3 = y_1^{-2}((x_1^2x_3^3)\circ\sigma_1)$.) Then $\mu_2(b) = 3$ and $\mu_{2H_1}(b) = 3/2$, so that $\nu_2(b) = 3 - 3/2 = 3/2$ and $\mathrm{inv}_{1\frac{1}{2}}(b) = (2, 0; 3/2)$. $D_2(b) = y_1^{3/2}$, so that $\mathscr{G}_2(b) = \{(y_2^3, 3)\}$, which is equivalent to $\{(y_2, 1)\}$. $(N_1(b), \mathscr{G}_2(b), \mathscr{E}_1(b))$ is a presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at b; therefore, $S_{\mathrm{inv}_{1\frac{1}{2}}}(b) = \{y_2 = y_3 = 0\}$. Repeating the procedure: $E^2(b) = \{H_1\}$, $\mathrm{inv}_2(b) = (2, 0; 3/2, 1)$ and inv_2 is presented at b by $(N_1(b), \mathscr{F}_2(b), \mathscr{E}_2(b) = \emptyset)$, where $\mathscr{F}_2(b) = \{(y_1, 1), (y_2, 1)\}$. Finally, $\mathrm{inv}_X(b) = (2, 0; 3/2, 1; 1, 0; \infty)$ and $S_{\mathrm{inv}_X}(b) = S_{\mathrm{inv}_2}(b) = \{y_1 = y_2 = y_3 = 0\} = \{b\}$. We let σ_2 be the blowing-up with centre $C_1 = \{b\}$. $\sigma_2^{-1}(U_1)$ is covered by 3 coordinate charts

$U_{1i} = \sigma_2^{-1}(U_1)\backslash\{y_i = 0\}'$, $i = 1,2,3$; $\sigma_2|U_{12}$ can be written $y_1 = z_1z_2$, $y_2 = z_2$, $y_3 = z_2z_3$.

Year two. Let X_2 denote the strict transform of X_1; in particular, $X_2 \cap U_{12} = V(g_2)$, where $g_2 = z_2^{-2}g_1\circ\sigma_2 = z_3^2 - z_1^3z_2^4$. Let $c = 0$ in U_{12}. Now, $E(c) = \{H_1, H_2\}$, where $H_1 = \{y_1 = 0\}' = \{z_1 = 0\}$ and $H_2 = \sigma_2^{-1}(b) = \{z_2 = 0\}$. Then $\nu_1(c) = 2 = \nu_1(a)$, so that $E^1(c) = \emptyset$, $s_1(c) = 0$, and $\mathscr{E}_1(c) = E(c)$. We take $\mathscr{F}_1(c) = \mathscr{G}_1(c) = \{(g_2, 2)\}$, $N_1(c) = \{z_3 = 0\}$, and $\mathscr{H}_1(c) = \{(z_1^3z_2^4, 2)\}$. Then $\mu_2(c) = 7/2$ and $D_2(c) = z_1^{3/2}z_2^2$, so that $\nu_2(c) = 0$ and $\mathrm{inv}_X(c) = \mathrm{inv}_{1\frac{1}{2}}(c) = (2,0;0)$. $(N_1(c), \mathscr{H}_1(c), \mathscr{E}_1(c))$ is a presentation of inv_1 (or of $\mathrm{inv}_{1/2}$) at c, and $\big(N_1(c), \mathscr{G}_2(c), \mathscr{E}_1(c)\big)$, where $\mathscr{G}_2(c) = \{\big(D_2(c), 1\big)\}$, is a presentation of $\mathrm{inv}_X = \mathrm{inv}_{1\frac{1}{2}}$ at c. $S_{\mathrm{inv}_X}(c) = S_{\mathrm{inv}_{1\frac{1}{2}}}(c)$ is the union of the z_2- and z_1-axes; $S_{\mathrm{inv}_X}(c) \cap H_1 = z_2$-axis and $S_{\mathrm{inv}_X}(c) \cap H_2 = z_1$-axis. In the lexicographic ordering of the set of subsets of $E(c)$ (given by 1.16), $\{H_1\} = (1,0) > (0,1) = \{H_2\}$, so that $J(c) = \{H_1\}$ and $\mathrm{inv}_X^e(c) = (\mathrm{inv}_X(c); \{H_1\})$. In other words, although (for property 1.14(4)) we could choose either component of $S_{\mathrm{inv}_X}(c)$ as centre of blowing-up, for the purpose of canonical desingularization we choose $C_2 = z_2$-axis. Let σ_3 be the blowing-up with centre C_2. $\sigma_3^{-1}(U_{12})$ is covered by 2 coordinate charts U_{12i}, where $U_{12i} = \sigma_3^{-1}(U_{12})\backslash\{z_i = 0\}'$, $i = 1,3$; $\sigma_3|U_{121}$ can be written $z_1 = w_1$, $z_2 = w_2$, $z_3 = w_1w_3$.

Year three. Let X_3 be the strict transform of X_2; in particular, $X_3 \cap U_{121} = V(g_3)$, where $g_3 = w_3^2 - w_1w_2^4$. Let $d = 0$ in U_{121}, so that $E(d) = \{H_2, H_3\}$, where $H_2 = \{z_2 = 0\}' = \{w_2 = 0\}$ and $H_3 = \sigma_3^{-1}(C_2) = \{w_1 = 0\}$. Then $\nu_1(d) = 2$, so that $E^1(d) = \emptyset$, $s_1(d) = 0$, and $\mathscr{E}_1(d) = E(d)$. We take $N_1(d) = \{w_3 = 0\}$ (still the strict transform of $N_1(a) = \{x_3 = 0\}$) and $\mathscr{H}_1(d) = \{(w_1w_2^4, 2)\}$. Then $\mu_2(d) = 5/2$ and $D_2(d) = w_1^{1/2}w_2^2$, so that $\nu_2(d) = 0$ and $\mathrm{inv}_X(d) = \mathrm{inv}_{1\frac{1}{2}}(d) = (2,0;0)$. Again, $\big(N_1(d), \mathscr{H}_1(d), \mathscr{E}_1(d)\big)$ is a presentation of inv_1 (or of $\mathrm{inv}_{1/2}$) at d, and $\big(N_1(d), \mathscr{G}_2(d), \mathscr{E}_1(d)\big)$, where $\mathscr{G}_2(d) = \{(D_2(d), 1)\}$ is a presentation of $\mathrm{inv}_X = \mathrm{inv}_{1\frac{1}{2}}$ at d. Therefore, $S_{\mathrm{inv}_X}(d) = S_{\mathrm{inv}_{1\frac{1}{2}}}(d) = \{w_2 = w_3 = 0\}$. We let σ_4 be the blowing-up with centre $C_3 = w_1$-axis. Note that $\mathrm{inv}_X(d) = \mathrm{inv}_X(c)$, but $\mu_X(d) = \mu_2(d) = 5/2 < 7/2 = \mu_2(c) = \mu_X(c)$ (as predicted by Theorem 1.14 (4)). $\sigma_4^{-1}(U_{121})$ is covered by 2 coordinate charts U_{121i}, where $U_{121i} = \sigma_4^{-1}(U_{121})\backslash\{w_i = 0\}'$, $i = 2,3$; $\sigma_4|U_{1212}$ can be written $w_1 = v_1$, $w_2 = v_2$, $w_3 = v_2v_3$.

Year four. Let X_4 be the strict transform of X_3; thus $X_4 \cap U_{1212} = V(g_4)$, where $g_4 = v_3^2 - v_1v_2^2$. Let $e = 0$ in U_{1212}, so that $E(e) = \{H_3, H_4\}$, where $H_3 = \{w_1 = 0\}' = \{v_1 = 0\}$ and $H_4 = \sigma_4^{-1}(C_3) = \{v_2 = 0\}$. Then $\nu_1(e) = 2$, so that $E^1(e) = \emptyset$, $s_1(e) = 0$, $\mathscr{E}_1(e) = E(e)$. As above, we get $\mu_2(e) = 3/2$ and $D_2(e) = v_1^{1/2}v_2$, so that $\mathrm{inv}_X(e) = (2,0;0)$. inv_X is presented at e by $\big(N_1(e), \mathscr{G}_2(e), \mathscr{E}_1(e)\big)$, where $N_1(e) = \{v_3 = 0\}$ and $\mathscr{G}_2(e) = \{\big(D_2(e), 1\big)\}$. Therefore, $S_{\mathrm{inv}_X}(e) = \{v_2 = v_3 = 0\}$. It is easy to see that, if we blow up with centre $C_4 = S_{\mathrm{inv}_X}(e)$, then the multiplicity of the strict transform decreases; in fact, the strict transform X_5 is non-singular.

Example 2.2. Consider $X = \{x_3^2 - x_1 x_2^2 = 0\}$ – the hypersurface in year four above – but without a history of blowings-up; i.e., $E(\cdot) = \emptyset$. Let $a = 0$. In this case, $\mathrm{inv}_{1\frac{1}{2}}(a) = (2, 0; 3/2)$ (cf. year zero above), and we can take $N_1(a) = \{x_3 = 0\}$, $\mathscr{H}_1(a) = \{(x_1 x_2^2, 2)\}$ and $\mathscr{G}_2(a) = \{(x_1 x_2^2, 3)\}$; $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_1(a) = \emptyset\big)$ is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a, and we get an equivalent presentation by replacing $\mathscr{G}_2(a)$ with $\{(x_1, 1), (x_2, 1)\}$. Therefore, $\mathrm{inv}_X(a) = (2, 0; 3/2, 0; 1, 0; \infty)$ (as in year zero above) As centre of blowing up we would choose $C = S_{\mathrm{inv}_X}(a) = \{a\}$ – not the x_1-axis as in year four of 2.1, although the singularity is the same!

Example 2.3. Consider the hypersurface $X = V(g) \subset \underline{k}^3$, where $g(x) = x_3^3 - x_1 x_2$.

Year zero. Let $a = 0$. Then $\nu_1(a) = \mu_a(g) = 2$ and $\mathrm{Sing} X = \{a\}$, so that $S_{\mathrm{inv}_X}(a) = \{a\}$. We therefore let $\sigma_1 : M_1 \to M_0 = \underline{k}^3$ be the blowing-up with centre $C_0 = \{a\}$. M_1 is covered by 3 coordinate charts $U_i = M_1 \backslash \{x_i = 0\}'$, where $\{x_i = 0\}'$ is the strict transform of $\{x_i = 0\}$, $i = 1, 2, 3$; $\sigma_1 | U_3$ can be written $x_1 = y_1 y_3$, $x_2 = y_2 y_3$, $x_3 = y_3$.

Year one. Let X_1 denote the strict transform of X by σ_1; then $X_1 \cap U_3 = V(g_1)$, where $g_1 = y_3^{-2} g \circ \sigma_1 = y_3 - y_1 y_2$. Let $b = 0$ in U_3. Then $\nu_1(b) = 1 < 2 = \nu_1(a)$; therefore $E^1(b) = E(b) = \{H_1\}$, where $H_1 = \sigma_1^{-1}(a) = \{y_3 = 0\}$, so that $s_1(b) = 1$ and $\mathscr{E}_1(b) = \emptyset$. We can take $\mathscr{F}_1(b) = \{(g_1, 1), (y_3, 1)\}$, $N_1(b) = \{y_3 = 0\}$ and $\mathscr{H}_1(b) = \{(y_1 y_2, 1)\}$. Then $\mu_2(b) = 2 = \nu_2(b)$, $\mathrm{inv}_{1\frac{1}{2}}(b) = (1, 1; 2)$ and $\mathscr{F}_2(b) = \mathscr{G}_2(b) = \{y_1 y_2, 2)\}$, which is equivalent to $\{(y_1, 1), (y_2, 1)\}$. It follows that $\mathrm{inv}_X(b) = (1, 1; 2, 0; 1, 0; \infty)$ and $S_{\mathrm{inv}_X}(b) = S_{\mathrm{inv}_{1\frac{1}{2}}}(b) = \{b\}$. Let σ_2 be the blowing-up with centre $C_1 = \{b\}$. $\sigma_2^{-1}(U_3)$ is covered by 3 coordinate charts $U_{3i} = \sigma_2^{-1}(U_3) \backslash \{y_i = 0\}'$, $i = 1, 2, 3$; $\sigma_2 | U_{31}$ can be written $y_1 = z_1$, $y_2 = z_1 z_2$, $y_3 = z_1 z_3$.

Year two. Let X_2 be the strict transform of X_1; in particular, $X_2 \cap U_{31} = V(g_2)$, where $g_2 = z_1^{-1} g_1 \circ \sigma_2 = z_3 - z_1 z_2$. Let $c = 0$ in U_{31}. Then $\nu_1(c) = 1 = \nu_1(b)$, and $E(c) = \{H_1, H_2\}$, where $H_1 = \{y_3 = 0\}' = \{z_3 = 0\}$ and $H_2 = \sigma_2^{-1}(b) = \{z_1 = 0\}$, so that $E^1(c) = \{H_1\}$, $s_1(c) = 1$ and $\mathscr{E}_1(c) = \{H_2\}$. We take $\mathscr{F}_1(c) = \{(g_2, 1), (z_3, 1)\}$, $N_1(c) = \{z_3 = 0\}$ and $\mathscr{H}_1(c) = \{(z_1 z_2, 1)\}$. Then $\mu_2(c) = 2$ and $D_2(c) = z_1$, so $\nu_2(c) = 1$ and $\mathrm{inv}_{1\frac{1}{2}}(c) = (1, 1; 1)$. Hence $E^2(c) = \{H_2\}$ and $\big(N_1(c), \mathscr{G}_2(c), \mathscr{E}_2(c)\big)$, where $\mathscr{G}_2(c) = \{(z_2, 1)\}$ and $\mathscr{E}_2(c) = \emptyset$, is a presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at c. It follows that $\mathrm{inv}_X(c) = (1, 1; 1, 1; 1, 0; \infty)$ and $S_{\mathrm{inv}_X}(c) = \{c\}$. Let σ_3 be the blowing-up with centre $C_2 = \{c\}$. $\sigma_3^{-1}(U_{31})$ is covered by 3 coordinate charts $U_{31i} = \sigma_3^{-1}(U_{31}) \backslash \{z_i = 0\}'$, $i = 1, 2, 3$; $\sigma_3 | U_{311}$ can be written $z_1 = w_1$, $z_2 = w_1 w_2$, $z_3 = w_1 w_3$.

Year three. Let X_3 be the strict transform of X_2; in particular, $X_3 \cap U_{311} = V(g_3)$, where $g_3 = w_3 - w_1 w_2$. Let $d = 0$ in U_{311}, so that $E(d) = \{H_1, H_3\}$, where $H_1 = \{w_3 = 0\}$ and $H_3 = \sigma_3^{-1}(c) = \{w_1 = 0\}$. Then $\nu_1(d) = 1$, $E^1(d) = \{H_1\}$, $s_1(d) = 1$ and $\mathscr{E}_1(d) = \{H_3\}$. We take $\mathscr{F}_1(d) = \{(g_3, 1), (w_3, 1)\}$, $N_1(d) = \{w_3 = 0\}$ and $\mathscr{H}_1(d) = \{(w_1 w_2, 1)\}$. Then $\mu_2(d) = 2$ and $D_2(d) = w_1$, so that $\nu_2(d) = 1$ and $\mathrm{inv}_{1\frac{1}{2}}(d) = (1, 1; 1)$. Hence $E^2(d) = \emptyset$, $\mathrm{inv}_2(d) = (1, 1; 1, 0)$ and

$\big(N_1(d), \mathscr{F}_2(d), \mathscr{E}_2(d)\big)$, where $\mathscr{F}_2(d) = \mathscr{G}_2(d) = \{(w_2, 1)\}$ and $\mathscr{E}_2(d) = \{H_3\}$, is a presentation of inv_2 at d. It follows that $\mathrm{inv}_X(d) = (1,1;1,0;\infty)$ and $S_{\mathrm{inv}_X}(d) = \{w_3 = 0, w_2 = 0\}$. In this chart U_{311}, X_3 is smooth and has only normal crossings with respect to the collection E_3 of all exceptional divisors at every point of $\{w_3 = w_2 = 0\}$ except $d = 0$ (cf. 1.7(3)).

An application: Łojasiewicz's inequalities. The fundamental inequalities of Łojasiewicz are immediate consequences of desingularization in the form of Theorem 1.10 (or 1.6 in the hypersurface case); in fact, we need only the following:

Theorem 2.4. *Let M be a manifold, and let $\mathscr{I} \subset \mathscr{O}_M$ denote a sheaf of (principal) ideals of finite type. Then there is a manifold M' and a proper surjective morphism $\varphi : M' \to M$ such that $\varphi^{-1}(\mathscr{I})$ is a normal-crossings divisor.*

Theorem 2.5. Inequality I. *Assume $\underline{k} = \mathbb{R}$ or $\mathbb{C}$. Let f and g be regular functions on a manifold M. (Recall that "regular" means "analytic" in the category of analytic spaces.) Suppose that K is a compact subset of M and that $\{x : g(x) = 0\} \subset \{x : f(x) = 0\}$ in a neighbourhood of K. Then there exist $c, \lambda > 0$ such that $|g(x)| \geq c|f(x)|^{\lambda}$ in a neighbourhood of K. The infimum of such λ is a positive rational.*

Proof. This is obvious if $f(x) \cdot g(x)$ has only normal crossings in a neighbourhood of K; in general, therefore, it follows from Theorem 2.4. □

Remark 2.6. We are assuming here that the category of spaces is from (0.2) (2) or (3). (If M has a quasi-compact underlying algebraic structure with respect to which f and g are regular, then λ can be chosen independent of K; there is an analogous remark concerning Inequalities II and III following.) The argument above allows us to conclude that, in any of the categories of (0.2), locally some power of $f \circ \varphi$ belongs to the ideal generated by $g \circ \varphi$; it follows that locally f belongs to the integral closure of the ideal generated by g, and the equation of integral dependence has degree bounded on K (cf. [LT]).

Theorem 2.7. Inequality II. *Let f be a regular function on an open subspace M of $\mathbb{R}^n$. Suppose that K is a compact subset of M, on which $\mathrm{grad}\, f(x) = 0$ only if $f(x) = 0$. Then there exist $c > 0$ and μ, $0 < \mu \leq 1$, such that $|\mathrm{grad}\, f(x)| \geq c|f(x)|^{1-\mu}$ in a neighbourhood of K. ($\mathrm{Sup}\,\mu$ is rational.)*

Proof. If $f(a) = 0$, then there is a neighbourhood of a in which $\mathrm{grad} f(x) = 0$ only if $f(x) = 0$. Let $g(x) = |\mathrm{grad} f(x)|^2 = \sum_{i=1}^{n} (\partial f / \partial x_i)^2$. (As for Inequality I) let $\varphi : M' \to M$ be a morphism given by 2.4 for the ideal generated by $f \cdot g$. We claim there is a neighbourhood of $\varphi^{-1}(K)$ in which $\varphi^*(f^2/g)$ is a regular function vanishing on $\{x : (f \circ \varphi)(x) = 0\}$: Consider a curve $\gamma : x = x(t)$ in M such that $\gamma \cap \{x : f(x) = 0\} = \{x(0)\}$ and γ is the image of a smooth curve in M' transverse to $\varphi^{-1}\big(\{x : g(x) = 0\}\big)$ at a smooth point of the latter. Let

$Q(t) = f\big(x(t)\big)$. Then $Q(t) \neq 0$ for $t \neq 0$, so $Q(t)$ has nonzero Taylor expansion at $t = 0$. Therefore, $Q(t)$ is divisible by $Q'(t) = df(dx/dt)$ and $Q(t)/Q'(t)$ vanishes at $t = 0$. Since $|Q'(t)|^2 \leq g\big(x(t)\big)|dx/dt|^2$, it follows that $Q(t)^2$ is divisible by $g\big(x(t)\big)$ and $f\big(x(t)\big)^2/g\big(x(t)\big)$ vanishes at $t = 0$. The claim follows.

From the claim, we conclude (as in 2.5) that there are $c, \mu > 0$ (where $\sup\mu$ is rational) such that $|f(x)|^{2\mu} \geq c^2 f(x)^2/g(x)$ in a neighbourhood of K. Clearly, $0 < \mu \leq 1$ ($\mu = 1$ if and only if $g(x)$ vanishes nowhere on K.) Thus, $|\mathrm{grad} f(x)| \geq c|f(x)|^{1-\mu}$. □

Theorem 2.8. Inequality III. *Let f be a regular function on an open subspace M of $\mathbb{R}^n$, and set $Z = \{x \in M : f(x) = 0\}$. Suppose K is a compact subset of M. Then there are $c > 0$ and $\nu \geq 1$ such that $|f(x)| \geq c d(x, Z)^\nu$ in a neighbourhood of K. ($d(\cdot, Z)$ is the distance to Z.) The infimum of such ν is rational.*

Proof. This follows from Inequality II: We can assume that $\mathrm{grad} f(x) = 0$ only if $f(x) = 0$, on K. We then claim that (even if f is merely $\mathscr{C}^1$ and) if $|\mathrm{grad} f(x)| \geq c|f(x)|^{1-\mu}$ in a neighbourhood U of K, where $0 < \mu \leq 1$, then $|f(x)|^\mu \geq \mu c d(x, Z)$ in some neighbourhood of K (cf. Łojasiewicz [Ł]): Consider a point $a \in U$ such that $f(a) \neq 0$. We can assume that $f(a) > 0$. (Otherwise, use $-f$.) Suppose that $x(t)$ is a solution of the equation $dx/dt = -\mathrm{grad} f(x)/|\mathrm{grad} f(x)|$ with $x(0) = a$. Write $Q(t) = f\big(x(t)\big)$. Then $Q'(t) = df(dx/dt) = -|\mathrm{grad} f\big(x(t)\big)| < 0$. Hence

$$\begin{aligned} \frac{f(a)^\mu}{\mu} \geq \frac{Q(0)^\mu - Q(t)^\mu}{\mu} &= -\frac{1}{\mu}\int_0^t \frac{d}{dt} Q(t)^\mu dt \\ &= -\int_0^t Q(t)^{\mu-1} Q'(t) dt \geq c \int_0^t dt = ct \ . \end{aligned}$$

It follows that the solution curve $x = x(t)$ tends to Z in a finite time t_0. Since $|dx/dt| = 1$, $t_0 \geq d(a, Z)$ and $f(a)^\mu \geq \mu c d(a, Z)$, as required. □

3. Basic notions

Definitions and notation. Let $X = (|X|, \mathscr{O}_X)$ denote a local-ringed space over $\underline{k}$. We call $|X|$ the *support* or *underlying topological space* of X. X is *smooth* if, for all $x \in |X|$, $\mathscr{O}_{X,x}$ is a regular local ring. A local-ringed space $Y = (|Y|, \mathscr{O}_Y)$ is a *closed subspace* of X if there is a sheaf of ideals $\mathscr{I}_Y$ of finite type in $\mathscr{O}_X$ such that $|Y| = \mathrm{supp}\, \mathscr{O}_X/\mathscr{I}_Y$ and $\mathscr{O}_Y$ is the restriction to $|Y|$ of $\mathscr{O}_X/\mathscr{I}_Y$. Y is an *open subspace* of X if $|Y|$ is an open subset of $|X|$ and $\mathscr{O}_Y = \mathscr{O}_X\big||Y|$.

Let $X = (|X|, \mathscr{O}_X)$ be a local-ringed space. Let $a \in |X|$. Suppose that $f \in \mathscr{O}_{X,a}$ (or that $f \in \mathscr{O}_X(U)$, where U is an open neighbourhood of a; we usually do not distinguish between $f \in \mathscr{O}_{X,a}$ and a representative in a suitable neighbourhood U). We define the *order* $\mu_a(f)$ *of* f *at* a as the largest $p \in \mathbb{N}$ such that $f \in \underline{m}_{X,a}^p$ (where $\underline{m}_{X,a}$ denotes the maximal ideal of $\mathscr{O}_{X,a}$). ($\mu_a f) = \infty$ if $f = 0$ in $\mathscr{O}_{X,a}$.)

Let C be a closed subspace of X, so that C is defined by a sheaf of ideals $\mathcal{I}_C \subset \mathcal{O}_X$ of finite type. We define the *order* $\mu_{C,a}(f)$ *of* f *along* C *at* a as the largest $p \in \mathbb{N}$ such that $f \in \mathcal{I}_{C,a}^p$.

Let $\varphi: X \to Y$ be a morphism of local-ringed spaces. Thus, if $a \in |X|$, φ induces local homomorphisms $\varphi_a^*: \mathcal{O}_{Y,\varphi(a)} \to \mathcal{O}_{X,a}$ (and $\widehat{\varphi}_a^*: \widehat{\mathcal{O}}_{Y,\varphi(a)} \to \widehat{\mathcal{O}}_{X,a}$ for the completions). If $g \in \mathcal{O}_{Y,\varphi(a)}$ (or $g \in \mathcal{O}_Y(V)$, where V is an open neighbourhood of $\varphi(a)$), then we will denote $\varphi_a^*(g)$ also by $g \circ \varphi_a$ or even by $g \circ \varphi$. (Similarly for $\widehat{\varphi}_a^*$.)

An element $f \in \mathcal{O}_X(U)$, where $U \subset |X|$ is open, will be called a *regular function* (*on* U). We will write X_a (respectively, f_a) for the germ at a of X (respectively, of f). If U is open in $|X|$ and $f_1, \ldots, f_\ell \in \mathcal{O}_X(U)$, then $V(f_1, \ldots, f_\ell)$ will denote the subspace of $X|U$ defined by the ideal subsheaf of $\mathcal{O}_X|U$ generated by the f_i.

Regular coordinate charts. If M is an analytic manifold, then a classical coordinate chart U is regular in the sense of (0.2). (Here $\mathcal{O}(U) = \mathcal{O}_M(U)$ means the ring of analytic functions on U.)

In this subsection, we show how to construct regular coordinate charts in the algebraic context. Consider a scheme of finite type over $\underline{k}$. Let $X = (|X|, \mathcal{O}_X)$ denote either the scheme itself, or the local-ringed space where $|X|$ is the set of $\underline{k}$-rational points of the scheme, with the induced Zariski topology, and $\mathcal{O}_X$ is the restriction to $|X|$ of the structure sheaf of the scheme. We will show that if $X = M$ is smooth, then M can be covered by coordinate charts as in (0.2). In the remainder of the article, we will adopt the convention that the residue field is $\underline{k}$ at every point (and write $\underline{k}^n$ rather than $\mathbb{A}^n$) in order to use a language common to schemes, analytic spaces, etc. But it will be clear from the construction of coordinate charts here (more precisely, from the fact that Taylor expansion commutes with differentiation and composition), that all of our constructions apply to points that are not necessarily $\underline{k}$-rational. (See Remark 3.8.)

Let $M = (|M|, \mathcal{O}_M)$. Each point a of M admits a Zariski-open neighbourhood U in which regular functions (elements of $\mathcal{O}(U) = \mathcal{O}_M(U)$) can be described as follows:

(3.1) (1) $U = V(p_1, \ldots, p_{N-n})$, where $N \geq n = \dim_a M$ and the $p_j \in \underline{k}[u, v]$ are polynomials in $(u, v) = (u_1, \ldots, u_n, v_1, \ldots, v_{N-n})$ such that $\det \partial p/\partial v$ vanishes nowhere on U (i.e., is invertible in the local ring of $\mathbb{A}^N$ at every point of U). ($\partial p/\partial v$ denotes the Jacobian matrix $\partial(p_1, \ldots, p_{N-n})/\partial(v_1, \ldots, v_{N-n})$.) We thus have a closed embedding $U \hookrightarrow \mathbb{A}^N$. (We say that the projection $(u, v) \mapsto u$ of $\mathbb{A}^N$ onto $\mathbb{A}^n$ induces an "étale covering" $U \to \mathbb{A}^n$.)

(2) Each element of $\mathcal{O}(U)$ is the restriction to U of a rational function $f = q/r$, where $q, r \in \underline{k}[u, v]$ and r vanishes nowhere on U.

If M is a scheme, $U = \operatorname{Spec} \underline{k}[u, v]/I$, where $I = (p_1, \ldots, p_{N-n})$ is the ideal generated by the p_i, and $\mathcal{O}(U)$ can be identified with $\underline{k}[u, v]/I$ (by the Nullstellensatz).

Definition 3.2. *A* **(regular) coordinate system** $(x_1, \ldots, x_n)$ *on a Zariski-open subset* U *of* $|M|$ *means an* n*-tuple of elements* $x_i \in \mathscr{O}(U)$ *satisfying the following condition: Let* $a \in U$. *Let* $a_i = x_i(a) \in \mathbb{F}_a$, $i = 1, \ldots, n$, *where* $\mathbb{F}_a$ *denotes the residue field* $\mathscr{O}_a/\underline{m}_a$. *(*$\underline{m}_a$ *is the maximal ideal of* $\mathscr{O}_a = \mathscr{O}_{M,a}$.*) If* $\Phi_i(z) \in \underline{k}[z]$ *denotes the minimal polynomial of* a_i *(i.e., the minimal monic relation for* a_i *with coefficients in* $\underline{k}$*),* $i = 1, \ldots, n$, *then the* $\Phi_i(x_i)$ *form a basis of* $\underline{m}_a/\underline{m}_a^2$ *over* $\mathbb{F}_a$.

In this case, $\dim \mathscr{O}_a = \dim_{\mathbb{F}_a} \underline{m}_a/\underline{m}_a^2$. If $\mathbb{F}_a = \underline{k}$ (i.e., if a is a $\underline{k}$*-rational point*) then $\Phi_i(x_i) = x_i - a_i$. In general, $\Phi_i(x_i) \sim x_i - a_i$ in the localization $\mathbb{F}_a[x_i]_{(a_i)}$. (We use $\sim$ to mean "= except for an invertible factor".)

In (3.1) above, the restrictions x_i to U of the u_i form a regular coordinate system $(x_1, \ldots, x_n)$. (The values of the coordinates may coincide at different points of U.)

Lemma 3.3. *Let* $a \in M$ *and let* $x_1, \ldots, x_n$ *denote regular functions on a neighbourhood of* a. *Then there is a Zariski-open neighbourhood* U *of* a *such that* $(x_1, \ldots, x_n)$ *is a regular coordinate system on* U *if and only if there is a closed embedding* $U \hookrightarrow \mathbb{A}^N$ *for some* N, *as in (3.1), and the* x_i *are the restrictions of the* u_i *to* U.

Proof. Let U be a Zariski-open neighbourhood of a such that U admits a closed embedding $U \hookrightarrow \mathbb{A}^N$ satisfying (3.1), and each $x_i \in \mathscr{O}(U)$; thus each x_i is the restriction to U of a function $q_i(u,v)/r_i(u,v)$, where $q_i, r_i \in \underline{k}[u,v]$ and $r_i(u,v)$ vanishes nowhere on U. Clearly, $(x_1, \ldots, x_n)$ forms a regular coordinate system on U if and only if the gradients of the q_i/r_i and the p_j are linearly independent at every point of U. Consider

$$\begin{array}{ccccc} U & \hookrightarrow & \mathbb{A}^{n+N} & & (y,u,v) \\ & \searrow & \downarrow & & \downarrow \\ & & \mathbb{A}^n & & y \end{array}$$

where $y = (y_1, \ldots, y_n)$ and U is embedded in $\mathbb{A}^{n+N}$ as $U = V\big(r_i(u,v)y_i - q_i(u,v), p_j(u,v)\big)$. Since each x_i is the restriction of y_i to U, $(x_1, \ldots, x_n)$ are regular coordinates if and only if $\det \partial(r_i y_i - q_i, p_j)/\partial(u,v)$ is a unit in $\mathscr{O}(U)$; the lemma follows. $\square$

We will call a Zariski-open subset U of $|M|$ which satisfies the conditions of Lemma 3.3 a *(regular) coordinate chart* with *(regular) coordinates* $x = (x_1, \ldots, x_n)$.

Definition 3.4. Taylor homomorphism. *Let* U *be a regular coordinate chart in* M, *with coordinates* $(x_1, \ldots, x_n)$. *For each* $a \in U$, *there is an injective* $\underline{k}$*-algebra homomorphism* $T_a\colon \mathscr{O}_{M,a} \to \mathbb{F}_a[[X]]$, $X = (X_1, \ldots, X_n)$, *that can be described as follows. Let* $p = (p_1, \ldots, p_{N-n})$ *(in the notation of (3.1)). By the formal implicit function theorem,* $p\big(u(a)+X,\ v(a)+V\big) = U(X,V)\big(V - \varphi(X)\big)$, *where* $\varphi(X) \in \mathbb{F}_a[[X]]^{N-n}$, $\varphi(0) = 0$, *and* $U(X,V)$ *is an invertible* $(N-n) \times (N-n)$ *matrix with entries in* $\mathbb{F}_a[[X,V]]$. *Let* $f \in \mathscr{O}_{M,a}$. *Then* f *is induced by an element* $F \in \underline{k}[u,v]_{(a)}$, *and* $(T_a f)(X) = F\big(u(a)+X,\ v(a)+\varphi(X)\big)$.

The *Taylor homomorphism* T_a induces an isomorphism $\widehat{\mathcal{O}}_{M,a} \to \mathbb{F}_a[[X]]$. Let $D^\alpha\colon \mathbb{F}_a[[X]] \to \mathbb{F}_a[[X]]$ denote the formal derivative $\partial^{|\alpha|}/\partial X^\alpha = \partial^{\alpha_1+\cdots+\alpha_n}/\partial X_1^{\alpha_1}\cdots\partial X_n^{\alpha_n}$, $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$.

Lemma 3.5. *Let U be a regular coordinate chart in M, with coordinates $x = (x_1,\ldots,x_n)$. Let $\alpha \in \mathbb{N}^n$. If $f \in \mathcal{O}(U)$, then there is (a unique) $f_\alpha \in \mathcal{O}(U)$ such that, for all $a \in U$,*

$$D^\alpha(T_a f)(X) = (T_a f_\alpha)(X) .$$

(We will write $f_\alpha = \partial^{|\alpha|}f/\partial x^\alpha$.) More precisely, if $\alpha = (j)$ for some j ((j) is the multiindex with 1 in the j'th place and 0 elsewhere; i.e., $D^\alpha = \partial/\partial X_j$) and if f is induced by $F = q/r$, where $q(u,v)$, $r(u,v) \in \underline{k}[u,v]$ (notation of (3.1)), then $f_{(j)}$ is induced by

$$F_{(j)} = \det\frac{\partial(F,p_1,\ldots,p_{N-n})}{\partial(u_j,v_1,\ldots,v_{N-n})}\Big/\det\frac{\partial(p_1,\ldots,p_{N-n})}{\partial(v_1,\ldots,v_{N-n})} .$$

Proof. It suffices to consider the case that $|\alpha| = 1$; i.e., $\alpha = (j)$, for some j. Let $a \in U$; say $\big(u(a), v(a)\big) = (0,0)$. From $(T_a f)(X) = F\big(X,\varphi(X)\big)$ (as in Definition 3.4) and from $p\big(X,\varphi(X)\big) = 0$, we obtain

$$\begin{aligned}\frac{\partial T_a f}{\partial X_j} &= \frac{\partial F}{\partial u_j}\big(X,\varphi(X)\big) + \frac{\partial F}{\partial v}\big(X,\varphi(X)\big)\cdot\frac{\partial\varphi}{\partial X_j} ,\\ 0 &= \frac{\partial p}{\partial u_j}\big(X,\varphi(X)\big) + \frac{\partial p}{\partial v}\big(X,\varphi(X)\big)\cdot\frac{\partial\varphi}{\partial X_j} .\end{aligned}$$

Thus, $\partial T_a f/\partial X_j = F_{(j)}\big(X,\varphi(X)\big)$, where

$$F_{(j)} = \frac{\left(\det\dfrac{\partial p}{\partial v}\right)\dfrac{\partial F}{\partial u_j} - \dfrac{\partial F}{\partial v}\cdot\left(\dfrac{\partial p}{\partial v}\right)^{\#}\cdot\dfrac{\partial p}{\partial u_j}}{\det\left(\dfrac{\partial p}{\partial v}\right)} .$$

($A^\#$ means the matrix such that $A\cdot A^\# = \det A\cdot I$.) The numerator here is the expression required in the lemma. $\square$

Remark 3.6. In Chapters II and III below, $\{x : \operatorname{inv}_X(x) \geq \operatorname{inv}_X(a)\}$ is constructively defined near any point a by combinations of derivatives of the original equations. It follows from Lemma 3.5 that this set and therefore the centre of our blowing-up are defined over the ground field $\underline{k}$ (even if a is not $\underline{k}$-rational).

Remark 3.7. Let $a \in U$. Suppose that $x_i(a) = 0$, $i = 1,\ldots,n$. (We use the notation above.) If $f \in \mathcal{O}(U)$ and $d \in \mathbb{N}$, then the Taylor expansion $(T_a f)(X)$ with respect to the regular coordinate system $x = (x_1,\ldots,x_n)$ can be written in a unique fashion as

$$(T_a f)(X) = c_0(\widetilde{X}) + c_1(\widetilde{X})X_n + \cdots + c_{d-1}(\widetilde{X})X_n^{d-1} + c_d(X)X_n^d ,$$

where $\widetilde{X} = (X_1, \ldots, X_{n-1})$. Of course, $\widetilde{x} = (x_1, \ldots, x_{n-1})$ forms a regular coordinate system on $N = V(x_n)$ and, for each $q = 0, \ldots, d-1$, $c_q(\widetilde{X})$ is the Taylor expansion at a of the regular function on N given by the restriction of $\dfrac{1}{q!}\dfrac{\partial^q f}{\partial x_n^q}$. Since the Taylor homomorphism is injective, we will write

$$f(x) \;=\; c_0(\widetilde{x}) + \cdots + c_{d-1}(\widetilde{x})x_n^{d-1} + c_d(x)x_n^d$$

for the Taylor expansion above, and we will identify each $c_q(\widetilde{x})$, when convenient, with the element of $\mathcal{O}_{N,a}$ induced by $\dfrac{1}{q!}\dfrac{\partial^q f}{\partial x_n^q}$. (In the case of analytic spaces, the preceding expression is just the usual convergent expansion with respect to x_n.)

Remark 3.8. All of the arguments in the paper involving $\widehat{\mathcal{O}}_{M,a}$ apply as they are written at an irrational point a provided that in the identification "$\widehat{\mathcal{O}}_{M,a} \cong \underline{k}[[X]]$", the field k is understood to be not the ground field but rather the residue field $\mathbb{F}_a$. If σ is a morphism and $\sigma(a') = a$, then $\mathbb{F}_a \subset \mathbb{F}_{a'}$ but they need not be equal. Nevertheless, the homomorphism of completions $\widehat{\sigma}^*_{a'}\colon \mathbb{F}_a[[X]] \to \mathbb{F}_{a'}[[Y]]$ induced by $\sigma^*_{a'}\colon \mathcal{O}_{M,a} \to \mathcal{O}_{M',a'}$ and the Taylor series homomorphisms in local coordinates, factors as

$$\mathbb{F}_a[[X]] \hookrightarrow \mathbb{F}_{a'}[[X]] \xrightarrow{\hat{\sigma}^*_{a'}} \mathbb{F}_{a'}[[Y]]\ .$$

In this context, "k" should be understood as $\mathbb{F}_{a'}$. (For example, in the proofs of Theorem 7.20 and 7.21 in the case of irrational points. Also in this way, the proof of Proposition 3.13 can be read as is in the case that $\mathbb{F}_a = \mathbb{F}_{a'}$; for the general case, see Remark 3.23.)

Properties of the category of spaces. Let $\mathcal{A}$ denote any of the (algebraic or analytic) categories of local-ringed spaces over $\underline{k}$ listed in (0.1) (1) and (2). Then $\mathcal{A}$ has the following essential features:

(3.9) (1) Let $X \in \mathcal{A}$. If Y is an open or a closed subspace of X, then $Y \in \mathcal{A}$. Locally, X is a closed subspace of a manifold $M \in \mathcal{A}$, where:

(2) A *manifold* $M = (|M|, \mathcal{O}_M)$ is a smooth space such that $|M|$ has a neighbourhood basis given by (the supports of) regular coordinate charts as in (0.2). (It follows that if X is a smooth subspace of a manifold M, then X is a manifold and is locally a coordinate subspace of a coordinate chart for M. In particular, every smooth space X is a manifold and is, therefore, locally pure-dimensional.)

(3) Let $X \in \mathcal{A}$. Then $\mathcal{O}_X$ is a coherent sheaf of rings and X is *locally Noetherian* in the sense of the following subsection.

(4) $\mathcal{A}$ is closed under blowing-up. (It follows that if $M \in \mathcal{A}$ is smooth, then a blowing-up $\sigma\colon M' \to M$ with smooth centre $C \subset M$ can be described locally as a quadratic transformation in regular coordinate charts.)

We recall that $\mathcal{O}_X$ is a coherent sheaf of rings if and only if every ideal of finite type in $\mathcal{O}_X$ is coherent. "Blowing-up" in (4) can be understood in terms of the universal mapping definition of Grothendieck (cf. [H1, Ch. 0, Sect. 2]). We do not need this definition (and therefore do not recall it); it follows from (3) above that if X is a closed subspace of a manifold M, then a blowing-up of X is given by the *strict transform* of X by a blowing-up of M. (See "Blowing up" and "The strict transform" below.)

The Zariski topology. Let $\mathscr{A}$ denote a category of local-ringed spaces over $\underline{k}$ (e.g., as in (0.1)). Let $X = (|X|, \mathscr{O}_X) \in \mathscr{A}$. A subset S of $|X|$ will be called a *Zariski-closed subset* of $|X|$ (or of X) if S is the support of a closed subspace of X (in $\mathscr{A}$). Suppose S and T are Zariski-closed subsets of $|X|$; say $S = \operatorname{supp} \mathscr{O}_X/\mathscr{I}$ and $T = \operatorname{supp} \mathscr{O}_X/\mathscr{J}$, where $\mathscr{I}, \mathscr{J} \subset \mathscr{O}_X$ are ideals of finite type that define closed subspaces in $\mathscr{A}$. Then $S \cap T = \operatorname{supp} \mathscr{O}_X/(\mathscr{I} + \mathscr{J})$ and $S \cup T = \operatorname{supp} \mathscr{O}_X/\mathscr{I} \cdot \mathscr{J}$ are Zariski-closed. A *Zariski-open subset* of $|X|$ (or of X) is the complement of a Zariski-closed subset. The Zariski-open subsets of $|X|$ define the *Zariski topology*. In general, the (original) topology of $|X|$ might be bigger than the Zariski topology (e.g., in the case of analytic spaces).

We say that X is *Noetherian* if every decreasing sequence of closed subspaces of X (in $\mathscr{A}$) stabilizes. We say that $|X|$ is *Noetherian* if every decreasing sequence of Zariski-closed subsets stabilizes. If X is Noetherian, then $|X|$ is Noetherian. We say that X (respectively, $|X|$) is *locally Noetherian* if every point of $|X|$ admits an open neighbourhood U (where U is the support of an open subspace in $\mathscr{A}$) such that every decreasing sequence of closed subspaces of X (respectively, Zariski-closed subsets of $|X|$) stabilizes on U. Clearly, if X (respectively, $|X|$) is locally Noetherian and $|X|$ is quasi-compact, then X (respectively, $|X|$) is Noetherian. (A real- or complex-analytic space X is Noetherian if and only if $|X|$ is compact.) If X is locally Noetherian, then the intersection of any family of closed subspaces of X is a subspace; hence the intersection of any family of Zariski-closed subsets of $|X|$ is Zariski-closed.

Lemma 3.10. *Suppose $|X|$ is Noetherian. Let Σ be a partially ordered set in which every decreasing sequence stabilizes. Let $\tau\colon |X| \to \Sigma$. Then the following are equivalent:*
(1) τ is upper-semicontinuous in the Zariski topology; i.e., each $a \in |X|$ admits a Zariski-open neighbourhood U such that $\tau(x) \le \tau(a)$ for all $x \in U$.
(2) τ takes only finitely many values and, for all $\sigma \in \Sigma$, $S_\sigma := \{x \in |X| : \tau(x) \ge \sigma\}$ is Zariski-closed.

Proof. Assume (1). Let $\sigma \in \Sigma$. Set $W = |X| \backslash S_\sigma$. If $a \in W$, then a has a Zariski-open neighbourhood U_a in which $\tau(x) \le \tau(a)$ (so that $\tau(x) \not\ge \sigma$); in particular, $U_a \subset W$. Thus $W = \bigcup_{a \in W} U_a$. Since $|X|$ is Noetherian, W is the union of finitely many U_a, so that W is Zariski-open, as required. It follows from the hypothesis on Σ that τ takes only finitely many values. Conversely, assume (2). Let $a \in |X|$. Set $U = \{x \in |X| : \tau(x) \le \tau(a)\}$. Then U is the complement of the finite union $\bigcup_{\sigma \not\le \tau(a)} S_\sigma$. □

Definition 3.11. *Let Σ denote a partially-ordered set. A function $\tau\colon |X| \to \Sigma$ is* **Zariski- semicontinuous** *if: (1) Locally, τ takes only finitely many values (locally* *with respect to open subspaces of X in $\mathscr{A}$). (2) For all $\sigma \in \Sigma$, $\{x \in |X| : \tau(x) \geq \sigma\}$ is Zariski-closed.*

The Hilbert-Samuel function $H_{X,\cdot}$ and therefore our invariant inv_X take values in partially-ordered sets satisfying the hypothesis of Lemma 3.10 (by [BM4, Theorem 5.2.1]; cf. Theorem 1.14). Our definition of inv_X in Sect. 6 shows that, when the given topology of $|X|$ differs from the Zariski topology, inv_X is semicontinuous in a sense that is (*a priori*) weaker than 3.11: (In the notation of Theorem 1.14), every point of $|M_j|$ has a coordinate neighbourhood U such that, for all $a \in U$, $V_a := \{x \in U : \mathrm{inv}_X(x) \leq \mathrm{inv}_X(a)\}$ is Zariski-open in $\left|M_j|U\right|$. If $|M_j|$ is locally Noetherian (or if X is a hypersurface), then, as in 3.10, there is a covering by coordinate charts U in which inv_X takes only finitely many values and (for any value ι), $\{x \in U : \mathrm{inv}_X(x) \geq \iota\}$ is Zariski-closed in $\left|M_j|U\right|$.

Of course, if $S \subset |X|$ and $|X|$ is covered by open subsets U such that each $S \cap U$ is the support of a smooth subspace of $X|U$, then S is globally the support of a smooth subspace of X. As a consequence, the centres of the blowings-up prescribed by our desingularization algorithm are always smooth *spaces.* Moreover, in the case of analytic spaces (for example), it follows from invariance of inv_X with respect to finite extension of the base field $\underline{k}$ that inv_X is actually Zariski-semicontinuous in the stronger sense.

Blowing-up. Let $\mathscr{A}$ be a category of local-ringed spaces over $\underline{k}$ as in (3.9). Let $M = (|M|, \mathscr{O}_M)$ be a smooth space in $\mathscr{A}$, and C a smooth subspace of M. Then C is covered by regular coordinate charts U of M, each of which has coordinates $x = (w, z)$, $w = (w_1, \ldots, w_{n-r})$, $z = (z_1, \ldots, z_r)$, in which $C \cap U = V(z) = V(z_1, \ldots, z_r)$.

Let σ: $M' = \mathrm{Bl}_C M \to M$ be the blowing-up of M with centre C. Let U be a regular coordinate chart as above, and let $U' = \sigma^{-1}(U)$. Then $U' \cong \{(a, \xi) \in U \times \mathbb{P}^{r-1} : z(a) \in \xi\}$, where $\mathbb{P}^{r-1}$ is the $(r-1)$-dimensional projective space of lines ξ through 0 in $\underline{k}^r$ (or $\mathbb{A}^r$); if we write $\xi \in \mathbb{P}^{r-1}$ as $\xi = [\xi_1, \ldots, \xi_r]$ in homogeneous coordinates, then

$$U' = \{(a, \xi) \in U \times \mathbb{P}^{r-1} : z_i(a)\xi_j = z_j(a)\xi_i,\ 1 \leq i,\ j \leq r\} .$$

Therefore, $U' = \bigcup_{i=1}^{r} U_i'$, where, for each i, $U_i' = \{(a, \xi) \in U' : \xi_i = 1\}$.

It follows that, for each i, U_i' is a regular coordinate chart with coordinates $x' = (w', z')$, $w' = (w_1', \ldots, w_{n-r}')$, $z' = (z_1', \ldots, z_r')$ given by $w'(a, \xi) = w(a)$, $z_i'(a, \xi) = z_i(a)$, and $z_j'(a, \xi) = z_j(a)/z_i(a)$ if $j \neq i$. In particular, suppose that $f \in \mathscr{O}_{M,a}$, where $a \in U$ and $w(a) = 0$, $z(a) = 0$; if $a' \in \sigma^{-1}(a) \cap U_i'$, then the Taylor expansion of $f \circ \sigma$ at a' is given by formal substitution of $w = w'$, $z_i = z_i'$ and $z_j = z_i'\left(z_j'(a') + z_j'\right)$, $j \neq i$, in the Taylor expansion of f at a.

Example 3.12. Let $M = (|M|, \mathscr{O}_M)$ be a smooth scheme of finite type over $\underline{k}$, and let U be a regular coordinate chart with coordinates $x = (x_1, \ldots, x_n)$ as in (3.1). As before, suppose that $x = (w, z)$ such that $C \cap U = V(z)$. Consider U_i', say for $i = 1$. (Using the notation of (3.1)) we have a commutative diagram

$$\begin{array}{ccc} U_1' & \hookrightarrow & \mathbb{A}^{N+(r-1)} \\ & \searrow & \downarrow \\ & & \mathbb{A}^n \end{array}$$

where U_1' is embedded in $\mathbb{A}^{N+(r-1)}$ as $V(p_1,\ldots,p_{N-n},\ q_j := u_{n-r+j} - u_{n-r+1}\xi_j,\ j = 2,\ldots,r)$ (in the affine coordinates $(u,v,\xi_2,\ldots,\xi_r)$ of $\mathbb{A}^{N+(r-1)}$) and the projection is given by $(u_1,\ldots,u_{n-r},u_{n-r+1},\xi_2,\ldots,\xi_r)$. $U_1' \to \underline{k}^n$ is an étale covering since

$$\det \frac{\partial(p,q_2,\ldots,q_r)}{\partial(v,u_{n-r+2},\ldots,u_n)} = \det \frac{\partial p}{\partial v} .$$

The strict transform. We use the notation of the preceding subsection. Let σ: $M' \to M$ be a blowing-up with smooth centre $C \subset M$, and let $H = \sigma^{-1}(C)$. Let X be a closed subspace of M. First suppose that X is a *hypersurface*; i.e., $\mathcal{I}_X$ is principal. Let $a \in M$ and let $f \in \mathcal{I}_{X,a}$ be a generator of $\mathcal{I}_{X,a}$. If $a' \in \sigma^{-1}(a)$, then we define $\mathcal{I}_{X',a'}$ as the principal ideal in $\mathcal{O}_{M',a'}$ generated by $f' = y_{\mathrm{exc}}^{-d} f \circ \sigma$, where y_{exc} denotes a generator of $\mathcal{I}_{H,a'}$ and $d = \mu_{C,a}(f)$. (Thus d is the largest power of y_{exc} to which $f \circ \sigma$ is divisible in $\mathcal{O}_{M',a'}$.) In this way we get a coherent sheaf of principal ideals $\mathcal{I}_{X'}$ in $\mathcal{O}_{M'}$; the *strict transform X' of X by σ* means the corresponding closed subspace of M'.

In local coordinates as above, suppose that $w(a) = 0$, $z(a) = 0$, and let $a' \in U_1'$. Then $y_{\mathrm{exc}} = z_1'$ and (the Taylor expansion at a' of) f' is given by

$$f'(w',z') = (z_1')^{-d} f\left(w', z_1', z_1'(\tilde{z}'(a) + \tilde{z}')\right) ,$$

where $\tilde{z}' = (z_2',\ldots,z_r')$. We will also call f' the "strict transform" of f by σ, although f' is, of course, only defined up to multiplication by an invertible factor.

The *strict transform by σ of* an arbitrary closed subspace X of M can be defined as the closed subspace X' of M' such that, locally at each $a' \in M'$, X' is the intersection of the strict transforms of all hypersurfaces containing X near $a = \sigma(a')$; i.e., $\mathcal{I}_{X',a'} \subset \mathcal{O}_{M',a'}$ is the ideal generated by the strict transforms f' of all $f \in \mathcal{I}_{X,a}$.

Proposition 3.13 (cf. [H1]). *Let $a' \in M'$. Then $\mathcal{I}_{X',a'}$ is the ideal $\{f \in \mathcal{O}_{M',a'} : y_{\mathrm{exc}}^k f \in \mathcal{I}_{\sigma^{-1}(X),a'},$ for some $k \in \mathbb{N}\}$. ($\mathcal{I}_{\sigma^{-1}(X),a'}$ is generated by $\sigma_{a'}^*(\mathcal{I}_{X,\sigma(a')})$.)*

We will give a simple proof of Proposition 3.13 below, as an application of the diagram of initial exponents. By Proposition 3.13, $\mathcal{I}_{X'} = \sum_k [\mathcal{I}_{\sigma^{-1}(X)} : y_{\mathrm{exc}}^k]$, so that $\mathcal{I}_{X'}$ is an ideal of finite type (since X is locally Noetherian).

Remark 3.14. (In a category satisfying the conditions (3.9)), we define the *reduced space* X_{red} corresponding to X using the coherent sheaf of ideals $\mathcal{I}_{X_{\mathrm{red}}} = \sqrt{\mathcal{I}_X}$. It is easy to see that if X' is the strict transform of X by a blowing-up, as above, then $(X')_{\mathrm{red}} = (X_{\mathrm{red}})'$.

Remark 3.15. Let X'' denote the smallest closed subspace of $\sigma^{-1}(X)$ containing $\sigma^{-1}(X)\backslash H$, where $H = \sigma^{-1}(C)$. (X'' exists by local Noetherianness.) Of course,

$X'' \subset X'$. In the case of schemes or analytic spaces over an algebraically closed field, $X'' = X'$. (A consequence of Hilbert's Nullstellensatz.) But $X'' \neq X'$ in general.

Example 3.16. Let $X \subset M = \mathbb{R}^2$ denote the real analytic subspace $x^4(x-1)^2+y^2 = 0$. Let $\sigma\colon M' \to M$ be the blowing-up with centre $\{0\}$. Then the strict transform $X' \subset U'$, where $U' \subset M'$ is a chart in which σ is given by $x = u$, $y = uv$. In U', X' is defined by $u^2(u-1)^2 + v^2 = 0$, so that $|X'| = \{(0,0),(1,0)\}$, but $|X''| = \{(1,0)\}$.

The diagram of initial exponents. The material in this subsection is needed only in Chapters III and IV (but we will also use the diagram to give a simple proof of Proposition 3.13 and to extend Remark 1.11 to the general case).

Let $\mathbb{K}$ be a field and $\mathbb{K}[[X]] = \mathbb{K}[[X_1,\dots,X_n]]$. If $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$, put $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The lexicographic ordering of $(n+1)$-tuples $(|\alpha|,\alpha_1,\dots,\alpha_n)$ induces a total ordering of $\mathbb{N}^n$. Let $F = \sum_{\alpha\in\mathbb{N}^n} F_\alpha X^\alpha \in \mathbb{K}[[X]]$, where $X^\alpha = X_1^{\alpha_1}\cdots X_n^{\alpha_n}$. Let $\operatorname{supp} F = \{\alpha : F_\alpha \neq 0\}$. The *initial exponent* $\exp F$ is the smallest element of $\operatorname{supp} F$. If $\alpha = \exp F$, then $F_\alpha x^\alpha$ is called the *initial monomial* $\operatorname{mon} F$ of F.

The following theorem of Hironaka [H1] (cf. [BM1, Theorem 6.2]) is a simple generalization of Euclidean division. Let $G^1,\dots,G^s \in \mathbb{K}[[X]]$, and let $\alpha^i = \exp G^i$, $i = 1,\dots,s$. We associate to $\alpha^1,\dots,\alpha^s$ a decomposition of $\mathbb{N}^n$: Set $\Delta_i = (\alpha^i + \mathbb{N}) - \bigcup_{j=1}^{i-1} \Delta_j$, $i = 1,\dots,s$, and put $\Box_0 = \mathbb{N}^n - \bigcup_{i=1}^{s} \Delta_i$. We also define $\Box_i \subset \mathbb{N}^n$ by $\Delta_i = \alpha^i + \Box_i$, $i = 1,\dots,s$.

Theorem 3.17. *For each $F \in \mathbb{K}[[X]]$, there are unique $Q_i \in \mathbb{K}[[X]]$, $i = 1,\dots,s$, and $R \in \mathbb{K}[[X]]$ such that $\operatorname{supp} Q_i \subset \Box_i$, $\operatorname{supp} R \subset \Box_0$, and $F = \sum_{i=1}^s Q_i G^i + R$.*

Remark 3.18. Let $\underline{m}$ denote the maximal ideal of $\mathbb{K}[[X]]$. In Theorem 3.17, if $k \in \mathbb{N}$ and $F \in \underline{m}^k$, then $R \in \underline{m}^k$ and each $Q_i \in \underline{m}^{k-|\alpha^i|}$ (where $\underline{m}^\ell$ means $\mathbb{K}[[X]]$ if $\ell \le 0$).

Let I be an ideal in $\mathbb{K}[[X]]$. The *diagram of initial exponents* $\mathfrak{N}(I) \subset \mathbb{N}^n$ is defined as $\mathfrak{N}(I) = \{\exp F : F \in I\}$. Clearly $\mathfrak{N}(I) + \mathbb{N}^n = \mathfrak{N}(I)$. Let $\mathscr{D}(n) = \{\mathfrak{N} \subset \mathbb{N}^n : \mathfrak{N} + \mathbb{N}^n = \mathfrak{N}\}$. If $\mathfrak{N} \in \mathscr{D}(n)$, then there is a smallest finite subset $\mathfrak{V}$ of $\mathfrak{N}$ such that $\mathfrak{N} = \mathfrak{V} + \mathbb{N}^n$; $\mathfrak{V} = \{\alpha \in \mathfrak{N} : \mathfrak{N}\backslash\{\alpha\} \in \mathscr{D}(n)\}$. We call $\mathfrak{V}$ the *vertices* of $\mathfrak{N}$.

Corollary 3.19. *Let α^i, $i = 1,\dots,s$, denote the vertices of $\mathfrak{N}(I)$. Choose $G^i \in I$ such that $\alpha^i = \exp G^i$, $i = 1,\dots,s$ (we say that G^i* represents *α^i), and let $\{\Delta_i,\Box_0\}$ denote the decomposition of $\mathbb{N}^n$ determined by the α^i, as above. Then:*

(1) $\mathfrak{N}(I) = \bigcup \Delta_i$ and the G^i generate I.

(2) There is a unique set of generators F^i of I, $i = 1,\dots,s$, such that, for each i, $\operatorname{supp}(F^i - x^{\alpha^i}) \subset \Box_0$; in particular, $\operatorname{mon} F^i = x^{\alpha^i}$.

We call $F^1, \ldots, F^s$ the *standard basis* of I (with respect to the given total ordering of $\mathbb{N}^n$). If $\mathfrak{N} \in \mathscr{D}(n)$, let $\mathbb{K}[[X]]^{\mathfrak{N}} = \{F \in \mathbb{K}[[X]] : \operatorname{supp} F \cap \mathfrak{N} = \emptyset$; i.e., $\operatorname{supp} F \subset \square_0\}$. Clearly, $\mathbb{K}[[X]]^{\mathfrak{N}}$ *is stable with respect to formal differentiation*.

Now let H_I denote the Hilbert-Samuel function of $\mathbb{K}[[X]]/I$; i.e., $H_I(k) = \dim_{\mathbb{K}} \mathbb{K}[[X]]/(I + \underline{m}^{k+1})$, $k \in \mathbb{N}$. By Remark 3.18 and Corollary 3.19, we have:

Corollary 3.20. *For every $k \in \mathbb{N}$, $H_I(k) = \#\{\alpha \in \mathbb{N}^n : \alpha \notin \mathfrak{N}(I)$ and $|\alpha| \leq k\}$. It follows that $H_I(k)$ coincides with a polynomial in k, for k large enough.*

Remark 3.21. The preceding definitions make sense and the results above (*except for* 3.18 and 3.20) hold, for any total ordering of $\mathbb{N}^n$ which is compatible with addition in the sense that: For any $\alpha, \beta, \gamma \in \mathbb{N}^n$, $\gamma \geq 0$, and $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$.

In order to prove Proposition 3.13, we will use the total ordering of $\mathbb{N}^n$ given by the lexicographic ordering of $(\alpha_1, |\alpha|, \alpha_2, \ldots, \alpha_n)$, $\alpha \in \mathbb{N}^n$. We then have:

Lemma 3.22. *Let I be an ideal in $\mathbb{K}[[Y]] = \mathbb{K}[[Y_1, \ldots, Y_n]]$, and let J denote the ideal $J = \{G(Y) \in \mathbb{K}[[Y]] : Y_1^k G(Y) \in I$, for some $k \in \mathbb{N}\}$. Suppose that $F_i(Y) \in I$, $i = 1, \ldots, s$, represent the vertices of $\mathfrak{N}(I)$; for each i, write $F_i(Y) = Y_1^{k_i} G_i(Y)$, where $G_i(Y)$ is not divisible by Y_1. Then J is generated by the G_i.*

Proof. This is an immediate consequence of the following variant of Remark 3.18 which holds for the given ordering of $\mathbb{N}^n$: In the formal division algorithm 3.17, if $F \in (Y_1)^k$, then $R \in (Y_1)^k$ and each $Q_i \in (Y_1)^{k-k_i}$. ((Y_1) denotes the ideal generated by Y_1.) $\square$

Proof of Proposition 3.13. We can choose coordinates at $a = \sigma(a')$ and a' so that $\widehat{\mathscr{O}}_{M,a} \cong \underline{k}[[X_1, \ldots, X_n]]$, $\widehat{\mathscr{O}}_{M',a'} \cong \underline{k}[[Y_1, \ldots, Y_n]]$ and $\widehat{\sigma}^*_{a'}: \widehat{\mathscr{O}}_{M,a} \to \widehat{\mathscr{O}}_{M',a'}$ has the form $X_\ell = Y_\ell$, $\ell = 1, \ldots, q$ (where $q \geq 1$), $X_\ell = Y_1(\eta_\ell + Y_\ell)$, $\ell = q+1, \ldots, n$. (See Remarks 3.8, 3.23.) Put $I = \widehat{\mathscr{I}}_{\sigma^{-1}(X),a'} \subset \underline{k}[[Y]]$ and $J = \{G(Y) \in \underline{k}[[Y]] : Y_1^k G(Y) \in I$, for some $k \in \mathbb{N}\}$. Suppose that $H_j(X)$, $j = 1, \ldots, r$, generate $\widehat{\mathscr{I}}_{X,a}$. We can find polynomials $P_{ij}(Y) \in \underline{k}[Y]$, $i = 1, \ldots, s$, $j = 1, \ldots, r$, such that the

$$F_i(Y) \ := \ \sum_j P_{ij}(Y)(H_j \circ \sigma)(Y) \in I$$

represent the vertices of $\mathfrak{N}(I)$. Each $F_i(Y)$ is the pullback by σ of

$$\sum_j P_{ij}\left(X_1, \ldots, X_q, \ \frac{X_{q+1}}{X_1} - \eta_{q+1}, \ldots, \frac{X_n}{X_1} - \eta_n\right) H_j(X) \ = \ \frac{1}{X_1^{q_i}} \sum_j Q_{ij}(X) H_j(X) \ ,$$

for some $q_i \in \mathbb{N}$, where the $Q_{ij} \in \underline{k}[X]$. Write $G_i(X) = \sum_j Q_{ij}(X)H_j(X)$, for each i. Thus each $F_i(Y) = (Y_1)^{-q_i}(G_i \circ \sigma)(Y)$. Write $(G_i \circ \sigma)(Y) = Y_1^{m_i} G_i'(Y)$, where G_i' is not divisible by Y_1, so that $m_i \geq q_i$. Then $F_i(Y) = Y_1^{m_i - q_i} G_i'(Y)$, for each i, where each $G'_i \in \widehat{\mathcal{I}}_{X',a'}$. But the G'_i generate J, by 3.22. (This formal argument suffices to prove the proposition because, for any ideal $\mathcal{J}$ in $\mathcal{O}_{M',a'}$, $\widehat{\mathcal{J}} \cap \mathcal{O}_{M',a'} = \mathcal{J}$.) $\square$

Remark 3.23. The proof above in the case $\mathbb{F}_a \subsetneqq \mathbb{F}_{a'}$ should be understood with the following modification: We can find polynomials $P_{ij} \in \widehat{\mathcal{O}}_{M,a}\left[\frac{X_{q+1}}{X_1}, \cdots, \frac{X_n}{X_1}\right]$, $i = 1, \ldots, s$, $j = 1, \ldots, r$, such that the $F_i(Y) := \sum_j (\widehat{\sigma}^*_{a'} P_{ij})(Y) \cdot (H_j \circ \sigma) \in I$ represent the vertices of $\mathfrak{N}(I)$. (This follows from Lemma 3.24 below, applied with $R = \widehat{\mathcal{O}}_{M,a}\left[\frac{X_{q+1}}{X_1}, \cdots, \frac{X_n}{X_1}\right]$; then $\widehat{\mathcal{O}}$ identifies with $\widehat{\mathcal{O}}_{M',a'} \cong \mathbb{F}_{a'}[[Y]]$, and $\mathcal{O}/\underline{m}^\ell_{\mathcal{O}} \to \widehat{\mathcal{O}}/\widehat{\underline{m}}^\ell_{\mathcal{O}}$ is an isomorphism for each ℓ.) Each $F_i(Y)$ is the pullback by σ of

$$\sum_j P_{ij}\left(X; \frac{X_{q+1}}{X_1}, \cdots, \frac{X_n}{X_1}\right) H_j(X) \;=\; (X_1)^{-q_i} \sum_j Q_{ij}(X) H_j(X)\ ,$$

for some $q_i \in \mathbb{N}$, where $Q_{ij} := X_1^{q_i} P_{ij} \in \widehat{\mathcal{O}}_{M,a} \cong \mathbb{F}_a[[X]]$. Etc.

Lemma 3.24. *If $\underline{m}$ is a maximal ideal in a domain R, and $\mathcal{O}$ denotes the localization of R at $\underline{m}$, then $R \to \mathcal{O}/\underline{m}^\ell_{\mathcal{O}}$ is surjective for any $\ell \in \mathbb{N}$, where $\underline{m}_{\mathcal{O}} = \underline{m} \cdot \mathcal{O}$.*

Proof. For any $Q \in R \backslash \underline{m}$, there is $\lambda \in R$ such that $\mu = 1 - \lambda Q \in \underline{m}$. Hence $(1 + \mu + \cdots + \mu^{\ell-1})\lambda Q = 1 \bmod \underline{m}^\ell$, which suffices. $\square$

Remark 3.25. The diagram of initial exponents can be used to generalize the geometric definition of inv_X in year zero given in Remark 1.11. Let $a \in M$ and let $(x_1, \ldots, x_n)$ denote a coordinate system at a, so that $\widehat{\mathcal{O}}_{M,a} \cong \underline{k}[[x_1, \ldots, x_n]]$ via the Taylor homomorphism. Let $w = (w_1, \ldots, w_n)$ be an n-tuple of positive real numbers ("weights" for the coordinates). For $f(x) = \sum f_\alpha x^\alpha \in \underline{k}[[x]]$, we define the *weighted order* $\mu_w(f) := \min\{\langle w, \alpha\rangle : \ f_\alpha \neq 0\}$ (where $\langle w, \alpha\rangle := \sum w_i \alpha_i$) and the *weighted initial exponent* $\exp_w(f) := \min\{\alpha : \ f_\alpha \neq 0\}$, where the $\alpha \in \mathbb{N}^n$ are totally ordered using lexicographic ordering of the sequences $(\langle w, \alpha\rangle, \alpha_1, \ldots, \alpha_n)$. Set $I = \widehat{\mathcal{I}}_{X,a}$. Write $\mathfrak{N}_w(I)$ for the (*weighted*) diagram of initial exponents $\{\exp_w(f) : \ f \in I\}$. If $\mathfrak{N} \in \mathcal{D}(n)$, we define the *essential variables* of $\mathfrak{N}$ as the indeterminates x_j which occur (to positive power) in some monomial x^α, where $\alpha \in \mathfrak{V}$ (the vertices of $\mathfrak{N}$).

For the given coordinate system $x = (x_1, \ldots, x_n)$, let $d(x)$ denote the supremum of n-tuples $(d_1, \ldots, d_n) \in (\mathbb{Q} \cup \{\infty\})^n$, ordered lexicographically, such that:

(1) $1 = d_1 \le d_2 \le \cdots \le d_n$;

(2) $x_1, \ldots, x_r$ are the essential variables of $\mathfrak{N}(I)$, for some r, and $d_1 = \cdots = d_r = 1$;

(3) $\mathfrak{N}(I) = \mathfrak{N}_w(I)$, where $w_j = 1/d_j$, $j = 1, \ldots, n$.

Here $\mathfrak{N}(I)$ denotes the diagram with respect to the standard ordering $(|\alpha|, \alpha_1, \ldots, \alpha_n)$ of $\mathbb{N}^n$. Set $d = \sup d(x)$ (sup over all coordinate systems), $d =$

$(d_1, \ldots, d_n)$. Then

$$\mathrm{inv}_X(a) = \left(H_{X,a}, 0; \frac{d_2}{d_1}, 0; \ldots; \frac{d_t}{d_{t-1}}, 0; \infty\right) ,$$

where d_t is the last finite d_i. As in 1.11, there is an explicit construction to obtain coordinates x such that $d(x) = d$, and there is a correspondence between the weighted initial ideals of I with respect to two coordinate systems that realize d.

Chapter II. The local construction; desingularization in the hypersurface case

The local construction that we use to define our invariant inv_X and establish its important properties (for example, invariance!) is presented in Sect. 4. Proofs of the nontrivial assertions are deferred to Sect. 5. At a first reading, one can skip the latter and go directly to Sect. 6 where, beginning with a presentation of $\mathrm{inv}_{1/2}$, we use the local construction recursively to define inv_X and a corresponding presentation, and we prove Theorem 1.14. In the hypersurface case, $\mathrm{inv}_{1/2}(a)$ (the order $\nu_{X,a}$ of X at a) admits a very simple presentation, so we complete the proof of desingularization for a hypersurface (and also Theorem 1.10; see Remark 1.18).

4. The local construction

Let M denote a manifold over $\underline{k}$. Consider the following data at a point $a \in M$:

(4.1) $N = N_p(a)$: a germ at a of a regular submanifold of codimension p;

$\mathscr{H}(a) = \{(h, \mu_h)\}$: a finite collection of pairs (h, μ_h), where each $h \in \mathscr{O}_{N,a}$ and each μ_h is a nonnegative rational number such that $\mu_h \le \mu_a(h)$;

$\mathscr{E}(a)$: a collection of smooth hypersurfaces $H \ni a$ such that N and $\mathscr{E}(a)$ simultaneously have only normal crossings, and $N \not\subset H$, for all $H \in \mathscr{E}(a)$.

We call $\left(N_p(a), \mathscr{H}(a), \mathscr{E}(a)\right)$ an *infinitesimal presentation* (of *codimension* p), and we define its *equimultiple locus* (as a germ at a)

$$S_{\mathscr{H}(a)} := \{x \in N : \ \mu_x(h) \ge \mu_h, \ \text{for all } (h, \mu_h) \in \mathscr{H}(a)\}.$$

Remark 4.2. We can assume (as we do below) that all "assigned multiplicities" $\mu_k \in \mathbb{N}$ because, given (4.1), there is an infinitesimal presentation which is equivalent (in the sense of Definition 4.6) and has integral μ_h (cf. Construction 4.23). But one can work with rational μ_h (as in [BM6]); this might be useful for efficiency of calculation.

By a *local blowing-up* σ: $M' \to M$ *over* a neighbourhood W of $a \in M$, we mean the composite of a blowing-up $M' = \mathrm{Bl}_C W \to W$ with smooth centre $C \subset W$, and the inclusion $W \hookrightarrow M$. (W can also be understood as a germ at a.)

Given an infinitesimal presentation (4.1), we consider morphisms $\sigma\colon M' \to M$ of three types:

(4.3) (i) *Admissible blowing-up.* $\sigma\colon M' = \mathrm{Bl}_C W \to W \hookrightarrow M$ is a local blowing-up over a neighbourhood W of a with smooth centre C such that $C \subset S_{\mathscr{H}(a)}$ and C, $\mathscr{E}(a)$ simultaneously have only normal crossings.

(ii) *Product with a line.* $\sigma\colon M' = W \times \underline{k} \to W \hookrightarrow M$ is a projection onto a neighbourhood W of a.

(iii) *Exceptional blowing-up.* $\sigma\colon M' = \mathrm{Bl}_C W \to W \hookrightarrow M$ is a local blowing-up with centre $C = H_0 \cap H_1$, where $H_0, H_1 \in \mathscr{E}(a)$.

We introduce a transform $\big(N_p(a'), \mathscr{H}(a'), \mathscr{E}(a')\big)$ of the infinitesimal presentation $\big(N_p(a), \mathscr{H}(a), \mathscr{E}(a)\big)$ by a morphism of each of these three types:

(4.4) (i) Let N' be the strict transform of $N = N_p(a)$ by σ, and let $a' \in \sigma^{-1}(a)$ such that $a' \in N'$ and $\mu_{a'}(h') \geq \mu_h$, for all $(h, \mu_h) \in \mathscr{H}(a)$, where $h' = y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma$. A transform of type (i) is defined provided such a' exists. We write σ also for the induced morphism $N' \to N$.) Set $N_p(a')$ = the germ of N' at a', $\mathscr{H}(a') = \{(h', \mu_h)\}$, and $\mathscr{E}(a') = \{\sigma^{-1}(C)\} \cup \{H' :\ H \in \mathscr{E}(a),\ a' \in H'\}$, where H' is the strict transform of H.

(ii) Let $a' = (a, 0) \in M \times \underline{k}$. Set $N' = N(a')$ = the germ of $\sigma^{-1}(N)$ at a', $\mathscr{H}(a') = \{(h \circ \sigma, \mu_h)\}$, and $\mathscr{E}(a') = \{W \times 0\} \cup \{H' = \sigma^{-1}(H) :\ H \in \mathscr{E}(a)\}$.

(iii) Let a' denote (the unique point of) $\sigma^{-1}(a) \cap H_1'$, where H' denotes the strict transform of H, for all $H \in \mathscr{E}(a)$. Set $N' = N_p(a')$ = the germ of $\sigma^{-1}(N)$ at a', $\mathscr{H}(a') = \{(h \circ \sigma, \mu_h)\}$, and $\mathscr{E}(a') = \{\sigma^{-1}(C)\} \cup \{H' :\ H \in \mathscr{E}(a),\ a' \in H'\}$.

It is clear that, in each case above, $\big(N_p(a'), \mathscr{H}(a'), \mathscr{E}(a')\big)$ is an infinitesimal presentation at a'. We will use the same notation $\big(N' = N_p(a'), \mathscr{H}(a'), \mathscr{E}(a')\big)$ for the transform of $\big(N_p(a), \mathscr{H}(a), \mathscr{E}(a)\big)$ by a sequence of morphisms of types (i), (ii), (iii).

Remark 4.5. A transformation of type (i) may be nontrivial even if $N = M$ and $\operatorname{codim} C = 1$, so that σ = identity. In (iii), if $H \in \mathscr{E}(a)$, $H \neq H_0, H_1$, then $C \not\subset H$ and $H' = \sigma^{-1}(H)$; likewise, N' is the strict transform of N (by the assumptions in (4.1)).

Definition 4.6. *Given $\mathscr{E}(a)$, we say that two infinitesimal presentations $\big(N = N_p(a), \mathscr{F}(a), \mathscr{E}(a)\big)$ and $\big(P = P_q(a), \mathscr{H}(a), \mathscr{E}(a)\big)$ are* **equivalent (with respect to transformations of types** *(i), (ii)* **and** *(iii)) if:*

(1) $S_{\mathscr{F}(a)} = S_{\mathscr{H}(a)}$.

(2) If σ is an admissible blowing-up (i) and $a' \in \sigma^{-1}(a)$, then $a' \in N'$ and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_f} f \circ \sigma) \geq \mu_f$, for all $(f, \mu_f) \in \mathscr{F}(a)$, if and only if $a' \in P'$ and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma) \geq \mu_h$, for all $(h, \mu_h) \in \mathscr{H}(a)$.

(3) After a transformation of type (i), (ii) or (iii), $\big(N', \mathscr{F}(a'), \mathscr{E}(a')\big)$ is equivalent to $\big(P', \mathscr{H}(a'), \mathscr{E}(a')\big)$.

Definition 4.6 makes sense recursively. We will write $\sim_{(\mathrm{i,ii,iii})}$ or merely $\sim$ (when there is no possibility of confusion) for this notion of equivalence. We will also write $\big(N, \mathscr{F}(a), \mathscr{E}(a)\big) \sim_{(\mathrm{i,ii})} \big(P, \mathscr{H}(a), \mathscr{E}(a)\big)$ when we have (1), (2) and the following weaker version of (3): After a transformation of type (i) or (ii), $\big(N', \mathcal{F}(a'), \mathcal{E}(a')\big) \sim_{(\mathrm{i,ii})} \big(P', \mathcal{H}(a'), \mathcal{E}(a')\big)$. We will use these ideas of equivalence only with $q = p$ or $q = p+1$. (In practice, a calculation of inv_X might considerably simplify when it is possible to replace an infinitesimal presentation by an equivalent one in higher codimension.)

Invariants of an infinitesimal presentation. We now introduce several important invariants of the equivalence classes of infinitesimal presentations of the same codimension. These invariants will be used to define the successive entries $\nu_{r+1}(a)$ of $\mathrm{inv}_X(a)$.

Definition 4.7. *We define $\mu(a) = \mu_{\mathcal{H}(a)}$, $1 \le \mu(a) \le \infty$, as*

$$\mu_{\mathcal{H}(a)} \;=\; \min_{(h,\mu_h)\in\mathcal{H}(a)} \frac{\mu_a(h)}{\mu_h} \;.$$

Proposition 4.8. *$\mu(a)$ depends only on the equivalence class of $\big(N, \mathcal{H}(a), \mathcal{E}(a)\big)$ with respect to transformations (i), (ii).*

In other words: Given $\mathcal{E}(a)$, let $\big(N^i, \mathcal{H}^i(a), \mathcal{E}(a)\big)$, $i = 1, 2$, be infinitesimal presentations of the same codimension p. Write $\mu^i(a) = \mu_{\mathcal{H}^i(a)}$, $i = 1, 2$. Then $\mu^1(a) = \mu^2(a)$ if the presentations are equivalent in the sense $\sim_{(\mathrm{i,ii})}$ (and therefore, of course, if the presentations are equivalent in the sense $\sim_{(\mathrm{i,ii,iii})}$). Proposition 4.8 will be proved in Sect. 5.

Definitions 4.9. *Suppose that $\mu(a) < \infty$. If $H \in \mathcal{E}(a)$, we define $\mu_H(a) = \mu_{\mathcal{H}(a),H}$ as*

$$\mu_{\mathcal{H}(a),H} \;=\; \min_{(h,\mu_h)\in\mathcal{H}(a)} \frac{\mu_{H,a}(h)}{\mu_h} \;,$$

where $\mu_{H,a}(h)$ denotes the order of h along $H \cap N$ at a. We define $\nu(a) = \nu_{\mathcal{H}(a)} \ge 0$ as

$$\nu(a) \;=\; \mu(a) - \sum_{H\in\mathcal{E}(a)} \mu_H(a) \;.$$

(We also put $\nu(a) = \infty$ if $\mu(a) = \infty$.)

Proposition 4.11 shows that the $\mu_H(a)$ and $\nu(a)$ are invariants of the equivalence class of our infinitesimal presentation $\big(N_p(a), \mathcal{H}(a), \mathcal{E}(a)\big)$ (where the codimension p is fixed) under an equivalence relation $\sim_*$ which is stronger than $\sim_{(\mathrm{i,ii})}$ but weaker than $\sim_{(\mathrm{i,ii,iii})}$. ("$\sim_*$ is weaker than $\sim_{(\mathrm{i,ii,iii})}$" means that the equivalence class of an infinitesimal presentation with respect to $\sim_{(\mathrm{i,ii,iii})}$ is a subset of that with respect to $\sim_*$.) We need $\sim_*$ because Construction 4.23 below survives transformations as allowed by $\sim_*$ (Proposition 4.24) but perhaps not an arbitrary sequence of transformations (i), (ii), (iii).

Definition 4.10. *We define $\sim_*$ by allowing in 4.6 only the transforms induced by certain sequences of morphisms of types (i), (ii), (iii); namely,*

$$\begin{array}{ccccccccc} \to & M_j & \xrightarrow{\sigma_j} & M_{j-1} & \to \cdots \xrightarrow{\sigma_{i+1}} & M_i & \to \cdots \to & M_0 = M \\ & \mathscr{H}(a_j) & & \mathscr{H}(a_{j-1}) & & \mathscr{H}(a_i) & & \mathscr{H}(a_0) = \mathscr{H}(a) \end{array}$$

where, if $\sigma_{i+1}, \ldots, \sigma_j$ are exceptional blowings-up (iii), then $i \geq 1$ and σ_i is of either type (iii) or (ii). In the latter case, σ_i: $M_i = W \times \underline{k} \to W \hookrightarrow M_{i-1}$, each σ_{k+1}, $k = i, \ldots, j-1$, is local blowing-up with centre $C_k = H_0^k \cap H_1^k$ where $H_0^k, H_1^k \in \mathscr{E}(a_k)$, $a_{k+1} = \sigma_{k+1}^{-1}(a_k) \cap H_1^{k+1}$, and we require that the H_0^k, H_1^k be determined by σ_i and some fixed $H \in \mathscr{E}(a_{i-1})$ inductively in the following way: $H_0^i = W \times 0$, $H_1^i = \sigma_i^{-1}(H)$, and, for $k = i+1, \ldots, j-1$, $H_0^k = \sigma_k^{-1}(C_{k-1})$, $H_1^k =$ the strict transform of H_1^{k-1} by σ_k.

Proposition 4.11. *Suppose that $\mu(a) < \infty$. Then $\mu_H(a)$, $H \in \mathscr{E}(a)$, and therefore also $\nu(a)$ depend only on the equivalence class of $\big(N, \mathscr{H}(a), \mathscr{E}(a)\big)$ with respect to $\sim_*$.*

The inductive construction. The successive entries $\nu_{r+1}(a)$ of $\mathrm{inv}_X(a)$ will be defined as $\nu_{\mathscr{H}_r(a)}$ for equivalence classes of certain infinitesimal presentations $\big(N_{p+r}(a), \mathscr{H}_r(a), \mathscr{E}_r(a)\big)$ constructed inductively in increasing codimension. Semicontinuity of $\mathrm{inv}_X(a)$ depends on choosing the local data in a "semicoherent" way; see 4.14. The following proposition (proved in Sect. 5) is the main tool in the induction on codimension (cf. 4.16).

Proposition 4.12. *Let $\big(N, \mathscr{F}(a), \mathscr{E}(a)\big) = \big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ denote an infinitesimal presentation (4.1) of codimension $p + r \geq 0$. Let $m = n - p - r$. Assume that $\mu_{\mathscr{F}(a)} = 1$ (i.e., there is $(f_*, \mu_{f_*}) \in \mathscr{F}(a)$ such that $\mu_a(f_*) = \mu_{f_*}$) and that there is a regular coordinate system $(x_1, \ldots, x_m)$ for N at a, in which $\partial^d f_* / \partial x_m^d$ is invertible at a (where $d = \mu_{f_*}$) and $\mathscr{E}(a) \cap N = \big\{\{x_i = 0\} : i \in I\big\}$, where $I \subset \{1, \ldots, m-1\}$. ($\mathscr{E}(a) \cap N$ means $\{H \cap N : H \in \mathscr{E}(a)\}$.) Then:*

(1) After any sequence of transformations (i), (ii), and (iii), $\mu_{\mathscr{F}(a')} = 1$. (In fact, $\mu_{a'}(f_*') = \mu_{f_*'} = d$, where f_*' denotes the transform of f_* in $\mathscr{F}(a')$.)

(2) Put $z = \partial^{d-1} f_* / \partial x_m^{d-1}$. Then $\mathscr{F}(a) \cup \{(z, 1)\} \sim_{(\mathrm{i,ii,iii})} \mathscr{F}(a)$.

(3) After any sequence of transformations (i), (ii), and (iii), $\{z' = 0\}$ and $\mathscr{E}(a') \cap N'$ simultaneously have only normal crossings, and $\{z' = 0\} \notin \mathscr{E}(a') \cap N'$.

Remarks 4.13. Condition (3) holds for any $z \in \mathscr{O}_{N,a}$ with $\mu_a(z) = 1$ which satisfies the analogous condition at a. The proof of 4.12 will show that, after an exceptional blowing-up (iii), $z' = z \circ \sigma$ coincides with the strict transform of z; likewise, if $(f, \mu_f) \in \mathscr{F}(a)$ and $\mu_a(f) = \mu_f$, then y_{exc} does not factor from $f' = f \circ \sigma$. Any infinitesimal presentation with $\mu_{\mathscr{F}(a)} = 1$ and $\mathscr{E}(a) = \emptyset$ satisfies the assumptions of 4.12 (cf. Example 4.16).

Definition and remarks 4.14. Let U be a regular coordinate chart in M. Suppose $a \in U$. We let $\mathscr{O}(U)_a$ denote the ring of quotients of elements of $\mathscr{O}(U) = \mathscr{O}_M(U)$ with denominators not vanishing at a. (In the case of schemes, $\mathscr{O}(U)_a = \mathscr{O}_{M,a}$.) If V is a Zariski-open subset of U, we will write $\mathscr{O}(U)_V$ to denote the ring of quotients of elements of $\mathcal{O}(U)$ with denominators vanishing nowhere in V.

The possibility of choosing local presentations of inv_X in a "semicoherent" way (see 6.4) will depend on the following observations: Assume in 4.12 that all of the given data is defined in $\mathcal{O}(U)_a$. In other words, $N = V(z_1, \ldots, z_{p+r})$, where $(z_1, \ldots, z_{p+r}$ are linearly independent $\mathrm{mod}\, \underline{m}_a^2$ and) each $z_j \in \mathcal{O}(U)_a$; also each f in $\mathcal{F}(a)$ and each coordinate function x_i on N is the restriction to N of an element of $\mathcal{O}(U)_a$. By Lemma 3.5, then $z = \partial^{d-1} f_* / \partial x_m^{d-1}$ is also the restriction to N of an element of $\mathcal{O}(U)_a$. It follows that there is a Zariski-open neighbourhood V of a in U such that: (1) Each $z_j \in \mathcal{O}(U)_V$, and $z_1, \ldots, z_{p+r}$ are linearly independent $\mathrm{mod}\, \underline{m}_x^2$, for all $x \in V(z_1, \ldots, z_{p+r}) \subset V$. (In particular, N extends to a regular submanifold of V.) (2) The f, the x_i, and z are all (restrictions to N of) elements of $\mathcal{O}(U)_V$, and $(x_1, \ldots, x_m)$ is a regular coordinate system on $N \subset V$. (3) $\mathcal{E}(a) \cap N = \big\{\{x_i = 0\} : \ i \in I\big\}$, and $\partial z / \partial x_m = \partial^d f_* / \partial x_m^d$ is invertible at every point of $S_{\mathcal{F}} = \{x \in N : \ \mu_x(f) \geq \mu_f, \ \text{for all } (f, \mu_f) \in \mathcal{F}(a)\}$. In particular, $\big(N_{p+r}(a), \mathcal{F}(a), \mathcal{E}(a)\big)$ induces an infinitesimal presentation $\big(N_{p+r}(x), \mathcal{F}(x), \mathcal{E}(x)\big)$ satisfying the hypotheses of 4.12, at each $x \in S_{\mathcal{F}}$.

Let σ: $M' \to M$ be a blowing-up with centre C. Assume that $C \cap U$ is a coordinate subspace of U and $C \cap V \subset S_{\mathcal{F}}$. It follows from the local-coordinate description of blowing-up (Sect. 3) that $\sigma^{-1}(U)$ is a union of finitely many regular coordinate charts U' of M' such that, if $a' \in \sigma^{-1}(a) \cap U'$ and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_f} f \circ \sigma) \geq \mu_f$, for all $(f, \mu_f) \in \mathcal{F}(a)$, then the transform $\big(N_{p+r}(a'), \mathcal{F}(a'), \mathcal{E}(a')\big)$ (type (i)) is given by data in $\mathcal{O}(U')_{a'}$.

Lemma 4.15. *Consider N and $\mathcal{E}(a)$ as in (4.1). Let $z \in \mathcal{O}_{N,a}$, $\mu_a(z) = 1$, and let C be (a germ at a of) a regular submanifold of N. Assume that C, $\mathcal{E}(a)$ and $V(z) \subset N$ simultaneously have only normal crossings, and $V(z) \notin \mathcal{E}(a) \cap N$. Then there is a regular coordinate system $x = (x_1, \ldots, x_m)$ for N at a, such that $x(a) = 0$ and: (1) $z = x_m$. (2) For all $H \in \mathcal{E}(a)$, $H \cap N = V(x_i)$, for some $i = 1, \ldots, m-1$. (3) $C = V(x_\ell, \ell \in J)$, for some $J \subset \{1, \ldots, m\}$.*

Moreover, if U is a regular coordinate chart, $a \in U$ and N, z, C are defined by functions in $\mathcal{O}(U)_a$, then there is a Zariski-open neighbourhood V of a in U so that the conclusion above holds in $N \cap V$, with each x_i (the restriction of) an element of $\mathcal{O}(U)_V$.

The proof is elementary. The example following shows the way we will obtain infinitesimal presentations satisfying the hypotheses of 4.12 in our inductive construction.

Example 4.16. Suppose $\big(N_{p+r}(a), \mathcal{G}_{r+1}(a), \mathcal{E}_{r+1}(a)\big)$ is an infinitesimal presentation at $a \in M$, with $\mathcal{E}_{r+1}(a) = \emptyset$ and $\mu_{\mathcal{G}_{r+1}(a)} = 1$. Let $\big(N_{p+r}(a'), \mathcal{G}_{r+1}(a'), \mathcal{E}_{r+1}(a')\big)$ be its transform by a finite sequence of admissible blowings-up as in (4.4)(i). Then $\big(N_{p+r}(a'), \ \mathcal{G}_{r+1}(a'), \mathcal{E}_{r+1}(a')\big)$ satisfies the assumptions of 4.12. (This follows from the proof of 4.12: Begin with suitable coordinates at a (where $\mathcal{E}_{r+1}(a) = \emptyset$) and transform.)

In particular, suppose that $\big(N_{p+r}(a), \mathscr{G}_{r+1}(a), \mathscr{E}_r(a)\big)$ is an infinitesimal presentation with $\mu_{\mathscr{G}_{r+1}(a)} = 1$. Set $E^{r+1}(a) = \mathscr{E}_r(a)$ and define a transformation of $E^{r+1}(a)$ by an admissible blowing-up (4.3)(i) as follows: $E^{r+1}(a') := \{H' : H \in E^{r+1}(a), a' \in H'\}$. Consider the transforms $\big(N_{p+r}(a'), \mathscr{G}_{r+1}(a'), \mathscr{E}_r(a')\big)$ and also $E^{r+1}(a') \subset \mathscr{E}_r(a')$ by a finite sequence of admissible blowings-up. Define $\mathscr{E}_{r+1}(a') := \mathscr{E}_r(a') - E^{r+1}(a')$ and $\mathscr{F}_{r+1}(a') := \mathscr{G}_{r+1}(a') \cup \big(E^{r+1}(a'), 1\big)$, where $\big(E^{r+1}(a'), 1\big) := \{(\ell_H, 1) : \; H \in E^{r+1}(a')\}$ and $\ell_H \in \mathscr{O}_{N',a'}$ denotes a generator of the ideal of $H \cap N' = H \cap N_{p+r}(a')$. Then $\big(N_{p+r}(a'), \mathscr{G}_{r+1}(a'), \mathscr{E}_{r+1}(a')\big)$ and therefore also $\big(N_{p+r}(a'), \mathscr{F}_{r+1}(a'), \mathscr{E}_{r+1}(a')\big)$ are infinitesimal presentations which satisfy the assumptions of Proposition 4.12.

Now suppose we have an infinitesimal presentation $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ of codimension $p + r$, which satisfies the following conditions from Proposition 4.12:

(4.17) (1) $\mu_{\mathscr{F}_{r+1}(a)} \;=\; 1$.

(2) There exists $z \in \mathscr{O}_{N,a}$, $N = N_{p+r}(a)$, such that $\mu_a(z) = 1$ and $\mathscr{F}_{r+1}(a) \cup \{(z, 1)\} \sim_{(\mathrm{i,ii,iii})} \mathscr{F}_{r+1}(a)$.

(3) $V(z)$ and $\mathscr{E}_{r+1}(a) \cap N_{p+r}(a)$ simultaneously have only normal crossings, and $V(z) \notin \mathscr{E}_{r+1}(a) \cap N_{p+r}(a)$.

We associate to $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ an equivalent infinitesimal presentation, in codimension $p + r + 1$:

Construction 4.18. Define $N_{p+r+1}(a) := V(z) \subset N_{p+r}(a)$. Then $N_{p+r+1}(a)$ and $\mathscr{E}_{r+1}(a)$ simultaneously have only normal crossings, and $N_{p+r+1}(a) \not\subset H$, for all $H \in \mathscr{E}_{r+1}(a)$. Choose regular coordinates $x = (x_1, \ldots, x_m = z)$ for $N = N_{p+r}(a)$ at a, as in 4.15. For each $(f, \mu_f) \in \mathscr{F}_{r+1}(a)$, consider the following formal expansion (cf. Remark 3.7):

$$f(x) \;=\; \sum_{0 \le q < \mu_f} c_{f,q}(\widetilde{x}) z^q + c_{f,\mu_f}(x) z^{\mu_f} \;,$$

where $\widetilde{x} = (x_1, \ldots, x_{m-1})$. Recall that each $c_{f,q}(\widetilde{x})$, $0 \le q < \mu_f$, is the element of $\mathscr{O}_{\widetilde{N},a}$, $\widetilde{N} = N_{p+r+1}(a)$, induced by $\dfrac{1}{q!}\dfrac{\partial^q f}{\partial x_m^q}$. Let $\mathscr{H}_{r+1}(a)$ denote the collection of pairs

$$\mathscr{H}_{r+1}(a) \;:=\; \{(c_{f,q}, \mu_f - q) : \; (f, \mu_f) \in \mathscr{F}_{r+1}(a), \; 0 \le q < \mu_f\} \;.$$

Proposition 4.19. $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big) \sim_{(\mathrm{i,ii,iii})} \big(N_{p+r+1}(a), \mathscr{H}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$. *Moreover, after any sequence of transformations of types (i), (ii) and (iii) (4.4), the transform* $\big(N_{p+r+1}(a'), \mathscr{H}_{r+1}(a'), \mathscr{E}_{r+1}(a')\big)$ *of* $\big(N_{p+r+1}(a), \mathscr{H}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ *is associated to* $\big(N_{p+r}(a'), \mathscr{F}_{r+1}(a'), \mathscr{E}_{r+1}(a')\big)$ *as in Construction 4.18.*

The proof is in Sect. 5. In general, of course, $\mu_{\mathscr{H}_{r+1}(a)} \neq 1$.

Remark 4.20. If $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ is defined by data in $\mathscr{O}(U)_a$, then so is $\big(N_{p+r+1}(a), \mathscr{H}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ (since each $c_{f,q}(\widetilde{x}) = \dfrac{1}{q!}\dfrac{\partial^q f}{\partial x_m^q}$ restricted to $\widetilde{N}$) .

We now put

$$\mu_{r+2}(a) \;=\; \mu_{\mathscr{H}_{r+1}(a)} \tag{4.21}$$

(cf. Definition 4.7); thus $1 \leq \mu_{r+2}(a) \leq \infty$. By Propositions 4.8 and 4.19, $\mu_{r+2}(a)$ depends only on the equivalence class of $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ with respect to $\sim_{(\mathrm{i,ii})}$ (and is therefore, of course, also an invariant of the equivalence class with respect to $\sim_*$).

If $\mu_{r+2}(a) < \infty$, then we set

$$\begin{aligned} \mu_{r+2,H}(a) \;&=\; \mu_{\mathscr{H}_{r+1}(a),H}\ , \qquad H \in \mathscr{E}_{r+1}(a)\ , \\ \nu_{r+2}(a) \;&=\; \mu_{r+2}(a) - \sum_{H \in \mathscr{E}_{r+1}(a)} \mu_{r+2,H}(a) \end{aligned} \tag{4.22}$$

(cf. Definitions 4.9); thus $\nu_{r+2}(a) \geq 0$. We also set $\nu_{r+2}(a) = \infty$ if $\mu_{r+2}(a) = \infty$. By Propositions 4.11 and 4.19, the $\mu_{r+2,H}(a)$ and $\nu_{r+2}(a)$ are invariants of (i.e., depend only on) the equivalence class of $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ with respect to $\sim_*$.

Construction 4.23. When $\nu_{r+2}(a) < \infty$, we now make the following construction beginning with our infinitesimal presentation $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ (satisfying the conditions (4.17) above). Define $\widetilde{N} = N_{p+r+1}(a)$ and $\mathscr{H}_{r+1}(a) = \{(h, \mu_h)\}$ as in Construction 4.18. We can assume that all μ_h are equal; say $\mu_h = d \in \mathbb{N}$, for all h. (For example, we can take $d = \max \mu_h!$ and replace each (h, μ_h) by $(h^{d/\mu_h}, d)$ to obtain a presentation which is equivalent with respect to $\sim_{(\mathrm{i,ii,iii})}$.) For each $H \in \mathscr{E}_{r+1}(a)$, we have $H \cap \widetilde{N} = V(x_i)$, for some $i = 1, \ldots, m-1$; say $x_i = x_H$. Set

$$D_{r+2}(a) \;=\; \prod_{H \in \mathscr{E}_{r+1}(a)} x_H^{\mu_{r+2,H}(a)}\ ;$$

thus $D = D_{r+2}(a)$ is a monomial in the coordinates $(x_1, \ldots, x_{m-1})$ of $\widetilde{N}$ with rational exponents. Clearly, D^d (which has exponents in $\mathbb{N}$) is the greatest common divisor of the h in $\mathscr{H}_{r+1}(a)$ that is a monomial in x_H, $H \in \mathscr{E}_{r+1}(a)$. Define $\mathscr{G}_{r+2}(a) = \{(g, \mu_g)\}$, where each $g \in \mathscr{O}_{\widetilde{N},a}$ and each $\mu_g \in \mathbb{N}$, as the collection of pairs $\{\big(g, d\nu_{r+2}(a)\big)$, for all $h = D^d g$ in $\mathscr{H}_{r+1}(a)$, together with $\big(D^d,\ \big(1 - \nu_{r+2}(a)\big)d\big)$ provided that $\nu_{r+2}(a) < 1\}$. ($\mathscr{G}_{r+2}(a) := \{(D^d, d)\}$ in the case $\nu_{r+2}(a) = 0$.) Then $\big(N_{p+r+1}(a), \mathscr{G}_{r+2}(a), \mathscr{E}_{r+1}(a)\big)$ is an infinitesimal presentation of codimension $p + r + 1$. If $\nu_{r+2}(a) > 0$, then $\mu_{\mathscr{G}_{r+2}(a)} = 1$. (The inductive construction terminates unless $0 < \nu_{r+2}(a) < \infty$.)

Of course, $S_{\mathscr{G}_{r+2}(a)} \subset S_{\mathscr{H}_{r+1}(a)} = S_{\mathscr{F}_{r+1}(a)}$. More precisely: $\big(N_{p+r+1}(a), \mathscr{H}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ induces a presentation $\big(N_{p+r+1}(x), \mathscr{H}_{r+1}(x), \mathscr{E}_{r+1}\ (x)\big)$ at x, for x in a neighbourhood of a in $S_{\mathscr{H}_{r+1}(a)}$. Clearly, $S_{\mathscr{G}_{r+2}(a)} = S_{\mathscr{F}_{r+1}(a),\nu_{r+2}(a)}$, where

$$S_{\mathscr{F}_{r+1}(a),\nu_{r+2}(a)} \;:=\; \{x \in S_{\mathscr{F}_{r+1}(a)}:\ \nu_{r+2}(x) = \nu_{r+2}(a)\}\ .$$

A local blowing-up which is admissible (i.e., a morphism of type (i)) for $\big(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\big)$ is admissible for $\big(N_{p+r+1}(a), \mathscr{G}_{r+2}(a), \mathscr{E}_{r+1}(a)\big)$ if and only if its centre $\subset S_{\mathscr{G}_{r+2}(a)}$. The following proposition is proved in Sect. 5.

Proposition 4.24. *Consider a sequence of transformations (i), (ii) and (iii) of* $\left(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\right)$ *as allowed by* $\sim_*$ *(Definition 4.10). If we assume recursively that the centres of the transformations of type (i) are admissible for the corresponding transforms of* $\left(N_{p+r+1}(a), \mathscr{G}_{r+2}(a), \mathscr{E}_{r+1}(a)\right)$*, then each succeeding transform* $\left(N_{p+r+1}(a'), \mathscr{G}_{r+2}(a'), \mathscr{E}_{r+1}(a')\right)$ *is associated to* $\left(N_{p+r}(a'), \mathscr{F}_{r+1}(a'), \mathscr{E}_{r+1}(a')\right)$ *as in 4.23.*

In particular, the equivalence class of $\left(N_{p+r+1}(a), \mathscr{G}_{r+2}(a), \mathscr{E}_{r+1}(a)\right)$ with respect to $\sim_*$ depends only that of $\left(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\right)$. It follows that if $\mathscr{E}_{r+2}(a) \subset \mathscr{E}_{r+1}(a)$, then the equivalence class of $\left(N_{p+r+1}(a), \mathscr{G}_{r+2}(a), \mathscr{E}_{r+2}(a)\right)$ with respect to $\sim_*$ depends only on that of $\left(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\right)$ (and on $\mathscr{E}_{r+2}(a)$) (cf. 4.16).

Remark 4.25. If $\left(N_{p+r}(a), \mathscr{F}_{r+1}(a), \mathscr{E}_{r+1}(a)\right)$ is given by data in $\mathscr{O}(U)_a$ (as in 4.14), then so is $\left(N_{p+r+1}(a), \mathscr{G}_{r+2}(a), \mathscr{E}_{r+1}(a)\right)$. This follows from 4.20 since passing from $\left(N_{p+r+1}(a), \mathscr{H}_{r+1}(a), \mathscr{E}_{r+1}(a)\right)$ to the latter involves only division by the regular function D^d.

5. Proofs

We prove Propositions 4.8, 4.11, 4.12, 4.19 and 4.24. We follow the notation of Sect. 4.

Lemma 5.1. *Let* $a \in M$ *and let* $\sigma\colon M' \to W \hookrightarrow M$ *be a local blowing-up over a neighbourhood* W *of* a *with smooth centre* $C \ni a$*. Let* $a' \in \sigma^{-1}(a)$*. Suppose that* $f \in \mathscr{O}_{M,a}$*. Set* $f' = y_{\mathrm{exc}}^{-\mu_{C,a}(f)} f \circ \sigma \in \mathscr{O}_{M',a'}$*, where* y_{exc} *denotes a generator of* $\mathscr{I}_{\sigma^{-1}(C),a'}$*. If* $\mu_{C,a}(f) = \mu_a(f)$*, then* $\mu_{a'}(f') \le \mu_a(f)$*.*

Proof. This is an elementary Taylor series computation (cf. [BM6, Lemma 2].) □

Proof of Proposition 4.8. Clearly, $\mu(a) = \infty$ if and only if $S_{\mathscr{H}(a)} = N_p(a)$; i.e., if and only if $S_{\mathscr{H}(a)}$ is (a germ of) a submanifold of codimension p in M.

Suppose that $\mu(a) < \infty$. Let $P_0 = W \times \underline{k} \to W \hookrightarrow M$ be a morphism of type (ii) at $a \in W$, and consider the transform $\left(N(c_0), \mathscr{H}(c_0), \mathscr{E}(c_0)\right)$ of $\left(N, \mathscr{H}(a), \mathscr{E}(a)\right)$ at $c_0 = (a, 0) \in P_0$ (cf. (4.3), (4.4)). Let γ_0 denote the arc $\gamma_0(t) = (a, t)$ in P_0. Consider the sequence of blowings-up

$$\longrightarrow P_{\beta+1} \xrightarrow{\sigma_{\beta+1}} P_\beta \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\sigma_1} P_0$$

with successive centres $c_\beta = \gamma_\beta(0)$, where $\gamma_{\beta+1}$ is defined inductively as the lifting of γ_β to $P_{\beta+1}$. (In other words, $\sigma_{\beta+1}^{-1}(c_\beta) \cap \Gamma_{\beta+1} = \{c_{\beta+1}\}$ for all $\beta \ge 0$, where $\Gamma_0 = \{a\} \times \underline{k}$ and $\Gamma_{\beta+1}$ is the strict transform of Γ_β by $\sigma_{\beta+1}$.) Then $\sigma_{\beta+1}$ induces a transformation of type (i), $\left(N(c_\beta), \mathscr{H}(c_\beta), \mathscr{E}(c_\beta)\right) \mapsto \left(N(c_{\beta+1}), \mathscr{H}(c_{\beta+1}), \mathscr{E}(c_{\beta+1})\right)$

(i.e., $\mu_{c_{\beta+1}}(y_{\mathrm{exc}}^{-\mu_h} h \circ \sigma_{\beta+1}) \geq \mu_h$, for all $(h, \mu_h) \in \mathscr{H}(c_\beta)$; cf. (4.4)(i)), successively for each $\beta \geq 0$: This follows from the transformation formula in power series (cf. proof of Lemma 5.2 below).

We introduce a subset S of $\mathbb{N} \times \mathbb{N}$ depending only on the equivalence class of $\big(N_p(a), \mathscr{H}(a), \mathscr{E}(a)\big)$ with respect to $\sim_{(\mathrm{i,ii})}$, as follows: First, say that $(\beta, 0) \in S$, $\beta \geq 1$, if after β blowings-up as above, there is (a germ of) a submanifold W_0 of codimension p in the hypersurface $\sigma_\beta^{-1}(c_{\beta-1})$ such that $W_0 \subset S_{\mathscr{H}(c_\beta)}$. (Then necessarily $W_0 = \sigma_\beta^{-1}(c_{\beta-1}) \cap N(c_\beta)$ and W_0, $\mathscr{E}(c_\beta)$ have only normal crossings.) In this case, we can blow up P_β locally with centre W_0. Put $Q_0 = P_\beta$, $d_0 = c_\beta$ and $\delta_0 = \gamma_\beta$. Inductively, say that $(\beta, \alpha) \in S$, $\alpha \geq 1$, if $(\beta, \alpha - 1) \in S$ and the following holds: Let τ_α: $Q_\alpha \to Q_{\alpha-1}$ be the local blowing-up with centre $W_{\alpha-1}$, and δ_α be the lifting of $\delta_{\alpha-1}$ by τ_α. Then:

(1) τ_α induces a transformation of type (i), $\big(N(d_{\alpha-1}), \mathscr{H}(d_{\alpha-1}), \mathscr{E}(d_{\alpha-1})\big) \mapsto \big(N(d_\alpha), \mathscr{H}(d_\alpha), \mathscr{E}(d_\alpha)\big)$; i.e., $\mu_{d_\alpha}(y_{\mathrm{exc}}^{-\mu_h} h \circ \tau_\alpha) \geq \mu_h$, for all $(h, \mu_h) \in \mathscr{H}(d_{\alpha-1})$.

(2) There exists a submanifold W_α of codimension p in the smooth hypersurface $\tau_\alpha^{-1}(W_{\alpha-1})$ such that $W_\alpha \subset S_{\mathscr{H}(d_\alpha)}$. (Necessarily, $W_\alpha = \tau_\alpha^{-1}(W_{\alpha-1}) \cap N(d_\alpha)$; clearly W_α and $\mathscr{E}(d_\alpha)$ simultaneously have only normal crossings.)

Since S depends only on the equivalence class of $\big(N_p(a), \mathscr{H}(a), \mathscr{E}(a)\big)$ with respect to $\sim_{(\mathrm{i,ii})}$, the proposition is a consequence of the following lemma. $\square$

Lemma 5.2. *$S = \emptyset$ if and only if $\mu(a) = 1$. If $S \neq \emptyset$, then*

$$S \;=\; \{(\beta, \alpha) \in \mathbb{N} \times \mathbb{N} :\; \beta\big(\mu(a) - 1\big) - \alpha \geq 1\} \;.$$

Lemma 5.2 specifies $\mu(a)$ uniquely; in the case that $1 < \mu(a) < \infty$, as $\mu(a) = 1 + \sup_{(\beta,\alpha)\in S}(\alpha + 1)/\beta$.

Proof of Lemma 5.2. We can choose a regular coordinate system $(x_1, \ldots, x_m)$ for $N = N_p(a)$ $(m = n - p)$ such that $a = 0$ and, for each $H \in \mathscr{E}(a)$, $H \cap N = V(x_i)$, for some $i = 1, \ldots, m$. We will write $(x_1, \ldots, x_m, x_0)$ for the corresponding regular coordinate system for $N(c_0) = N \times \underline{k}$. There is a regular coordinate system $(y_1, \ldots, y_m, y_0)$ for $N(c_1)$ in which σ_1: $N(c_1) \to N(c_0)$ is given by $x_0 = y_0$ and $x_\ell = y_0 y_\ell$, $\ell = 1, \ldots, m$. In these coordinates, $c_1 = 0$, $\Gamma_1 = V(y_1, \ldots, y_m)$, and

$$\mathscr{H}(c_1) \;=\; \{(h', \mu_{h'}) \;=\; (y_0^{-\mu_h} h \circ \sigma_1, \mu_h) :\; (h, \mu_h) \in \mathscr{H}(c_0)\} \;.$$

(Each $h = h(x_1, \ldots, x_m)$, independent of x_0.)

We can assume that all μ_h are equal; say $\mu_h = d \in \mathbb{N}$, for all $(h, \mu_h) \in \mathscr{H}(c_0)$. If $\mu(a) < \infty$, then the h' in $\mathscr{H}(c_1)$ admit $y_0^{(\mu(a)-1)d}$ as greatest common divisor which is a power of $y_0 = y_{\mathrm{exc}}$. Write $h' = y_0^{(\mu(a)-1)d}\widetilde{h}'$, for all h', and set $\widetilde{\mu}(c_1) = \min \mu_{c_1}(\widetilde{h}')/d$. Take h' such that $\widetilde{h}'$ realizes the min; it is clear from the formal expansions that the initial form of h' *equals* that of h, so $\widetilde{\mu}(c_1) = \mu(c_0) = \mu(a)$.

It follows that, after β blowings-up $\sigma_1, \ldots, \sigma_\beta$ as above, the transform $\mathscr{H}(c_\beta) = \{(h', d)\}$ of $\mathscr{H}(c_0)$ satisfies the following condition: There is a regular coordinate system $(y_1, \ldots, y_m, y_0)$ for $N(c_\beta)$ in which $\sigma_1 \circ \cdots \circ \sigma_\beta$ is given by $x_0 = y_0$ and $x_\ell = y_0^\beta y_\ell$, $\ell = 1, \dots, m$, $\Gamma_\beta = V(y_1, \dots, y_m)$, and each $h' = y_0^{\beta(\mu(a)-1)d} \widetilde{h}'$, where the $\widetilde{h}'$ do not admit $y_0 = y_{\mathrm{exc}}$ as common factor.

Let $W_0 = \sigma_\beta^{-1}(c_{\beta-1}) \cap N(c_\beta)$. Then $W_0 \subset S_{\mathscr{H}(c_\beta)}$ if and only if $\mu_{W_0, c_\beta}(h') \geq d$, for all h' in $\mathscr{H}(c_\beta)$. Since the $\widetilde{h}'$ do not admit y_0 as common factor, $W_0 \subset S_{\mathscr{H}(c_\beta)}$ if and only if $\beta\big(\mu(a) - 1\big) \geq 1$. In particular, $\mu(a) = 1$ if and only if $W_0 \not\subset S_{\mathscr{H}(c_\beta)}$ after any number β of blowings-up as described.

Now suppose that $W_0 \subset S_{\mathscr{H}(c_\beta)}$, and consider $\tau_1 \colon Q_1 \to Q_0 = P_\beta$ as in the proof of Proposition 4.8. There are regular coordinates $(z_1, \dots, z_m, z_0)$ for $N(d_1)$ in which $z_0 = z_{\mathrm{exc}}$ and $\tau_1 \colon N(d_1) \to N(d_0) = N(c_\beta)$ is given by the identity transformation $y_\ell = z_\ell$, $\ell = 0, \dots, m$. In the coordinates $(z_1, \dots, z_m, z_0)$, we have $d_1 = 0$,

$$\mathscr{H}(d_1) \;=\; \{(h', \mu_{h'}) \;=\; (z_0^{-d} h(z), d) :\; (h, \mu_h = d) \in \mathscr{H}(d_0) = \mathscr{H}(c_\beta)\}\;,$$

and each $\widetilde{h}' = \widetilde{h}(z)$. Thus $h' = z_0^{\beta(\mu(a)-1)d - d} \widetilde{h}'$, for all h' in $\mathscr{H}(d_1)$, where the $\widetilde{h}'$ do not admit z_0 as common factor. After α such blowings-up $\tau_1, \dots, \tau_\alpha$, $\mathscr{H}(d_\alpha) = \{(h', d)\}$, where each $h' = z_0^{\beta(\mu(a)-1)d - \alpha d} \widetilde{h}'$, and the $\widetilde{h}'$ do not admit z_0 as common factor. As above, $W_\alpha = \tau_\alpha^{-1}(W_{\alpha-1}) \cap N(d_\alpha) \subset S_{\mathscr{H}(d_\alpha)}$ if and only if $\beta\big(\mu(a) - 1\big) - \alpha \geq 1$. $\square$

Proof of Proposition 4.11. Let $H \in \mathscr{E}(a)$. Let $\sigma_0 \colon P_0 = W \times \underline{k} \to W \hookrightarrow M$ be a morphism of type (ii) at $a \in W$. Put $a_0 = (a, 0)$, $H_0^0 = W \times 0$, $H_1^0 = \sigma_0^{-1}(H)$. We follow σ_0 by a sequence of morphisms of type (iii) (exceptional blowings-up),

$$\longrightarrow P_{j+1} \xrightarrow{\sigma_{j+1}} P_j \longrightarrow \cdots \xrightarrow{\sigma_1} P_0 \xrightarrow{\sigma_0} M\;,$$

where each σ_{j+1}, $j \geq 0$, denotes the blowing-up with centre $C_j = H_0^j \cap H_1^j$, and $H_0^{j+1} = \sigma_{j+1}^{-1}(C_j)$, H_1^{j+1} = the strict transform of H_1^j. Let γ_0 denote the arc $\gamma_0(t) = (a, t)$ in P_0, and, for each j, take $a_{j+1} = \gamma_{j+1}(0)$, where γ_{j+1} is the lifting of γ_j by σ_{j+1}. The sequence of morphisms σ_j induces a sequence of transforms $\big(N(a_j), \mathscr{H}(a_j), \mathscr{E}(a_j)\big)$ of our infinitesimal presentation $\big(N(a), \mathscr{H}(a), \mathscr{E}(a)\big)$, as allowed in the definition of $\sim_*$ (4.10).

There is a regular coordinate system $(x_1, \dots, x_m)$ for $N = N_p(a)$ $(m = n - p)$ such that $a = 0$ and, for each $K \in \mathscr{E}(a)$, $K \cap N = V(x_i)$ for some $i = 1, \dots, m$ (we set $x_i = x_K$). Write $(x_1, \dots, x_m, x_0)$ for the corresponding regular coordinate system for $N(a_0) = N \times \underline{k}$. We can assume that $x_1 = x_H$. Then there is a regular coordinate system $(y_1, \dots, y_m, y_0)$ for $N(a_1)$ in which $\sigma_1 \colon N(a_1) \to N(a_0)$ is given by $x_0 = y_0$, $x_1 = y_0 y_1$, $x_\ell = y_\ell$, for $\ell = 2, \dots, m$, and in which $a_1 = 0$, $y_1 = y_H$ (y_H means $y_{H_1^1}$). Proceeding inductively, there are regular coordinates $(y_1, \dots, y_m, y_0)$ for each $N(a_j)$ in which $a_j = 0$ and $\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_j \colon N(a_j) \to N(a_0)$ is given by

$$x_0 = y_0\;, \quad x_H = x_1 = y_0^j y_1 = y_0^j y_H\;, \quad x_\ell = y_\ell\;, \;\; \ell = 2, \dots, m\;.$$

We can assume that all μ_h are equal; say $\mu_h = d \in \mathbb{N}$, for all $(h, \mu_h) \in \mathscr{H}(a)$. Set $D = \prod_{K \in \mathscr{E}(a)} x_K^{\mu_K(a)}$. Thus D^d is a monomial in $(x_1, \dots, x_m)$ with exponents in $\mathbb{N}$, and D^d is the greatest common divisor of the h in $\mathscr{H}(a)$ which is a monomial in x_K, $K \in \mathscr{E}(a)$. In particular, for some $h = D^d g$ in $\mathscr{H}(a)$, $g = g_H$ is not divisible by $x_1 = x_H$. Therefore, there exists $i \geq 1$ such that $\mu_{a_j}(g_H \circ \pi_j) = \mu_{a_i}(g_H \circ \pi_i)$ for all $j \geq i$, where $\pi_j := \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_j$. (We can simply take i to be the least order of a monomial not involving x_H in the Taylor expansion of g_H.)

On the other hand, for each $h = D^d g$ in $\mathscr{H}(a)$, $\mu_{a_j}(g \circ \pi_j)$ increases as $j \to \infty$ unless g is not divisible by x_H. Therefore, we can choose $h = D^d g_H$, as above, and i large enough so that we also have $\mu(a_j) = \mu_{a_j}(h \circ \pi_j)/d$, for all $j \geq i$. Clearly, if $j \geq i$, then $\mu_H(a) = \mu(a_{j+1}) - \mu(a_j)$, so the result follows from Proposition 4.8. □

Proof of Proposition 4.12. We can assume that $a = 0$ in the given coordinate system $(x_1, \ldots, x_m)$ for N (i.e., each $x_i(a) = 0$). For each $(f, \mu_f) \in \mathscr{F}(a)$, the Taylor expansion of f at a with respect to these coordinates can be written (cf. Remark 3.7) as

$$f(x) = \sum_{0 \leq q < \mu_f} c_{f,q}(\widetilde{x}) x_m^q + c_{f,\mu_f}(x) x_m^{\mu_f} \,, \tag{5.3}$$

where $\widetilde{x} := (x_1, \ldots, x_{m-1})$ and $\mu_a(c_{f,q}) \geq \mu_f - q$, $0 \leq q < \mu_f$. By hypothesis, $c_{f_*,\mu_{f_*}}$ is invertible.

Set $z = \partial^{d-1} f_* / \partial x_m^{d-1}$, where $d = \mu_{f_*}$. By the formal implicit function theorem,

$$z = u(x)\bigl(x_m - \varphi(\widetilde{x})\bigr) \,, \tag{5.4}$$

where u us a unit. We introduce a formal coordinate change,

$$x'_\ell = x_\ell, \ \ell = 1, \ldots, m-1, \quad x'_m = x_m - \varphi(\widetilde{x}) \,.$$

Note. Suppose that $g(x) = g(x_1, \ldots, x_m)$ is a formal power series. Write $g(x) = g'(x')$ in the new coordinates x'; i.e., $g'(x') = g\bigl(\widetilde{x}', x'_m + \varphi(\widetilde{x}')\bigr)$. Then, for all $q \in \mathbb{N}$, $\partial^q g'(x')/\partial {x'_m}^q = \partial^q g(x)/\partial x_m^q$. In particular, by (5.4),

$$\frac{\partial^{d-1} f'_*}{\partial {x'_m}^{d-1}} \sim x'_m \tag{5.5}$$

($\sim$ means = up to an invertible factor).

Therefore, after a *formal* coordinate change as above, we can assume (dropping primes) that in (5.3) above, $\mu_a(c_{f,q}) \geq \mu_f - q$, $0 \leq q < \mu_f$, $c_{f_*,d}$ is invertible, and $c_{f_*,d-1} = 0$ (by (5.5)); moreover, each $c_{f,q}$, $0 \leq q < \mu_f$, regarded as an element of $\widehat{\mathscr{O}}_{N,a}/(z)\widehat{\mathscr{O}}_{N,a}$, is (a germ at a of) a regular function on the regular submanifold $V(z) \subset N$. ($\widetilde{x} = (x_1, \ldots, x_{m-1})$ is a *regular* coordinate system for $V(z)$.)

In particular, $S_{\mathscr{F}(a)} \subset V(z)$. It follows that (as germs at a)

$$S_{\mathscr{F}(a)} = \{x \in V(z) : \mu_x(c_{f,q}) \geq \mu_f - q \,, \ 0 \leq q < \mu_f \,, \text{ for all } (f, \mu_f) \in \mathscr{F}(a)\} \,.$$

The effect of an admissible blowing-up. Let $\sigma\colon M' = \mathrm{Bl}_C W \to W \hookrightarrow M$ be a local blowing-up with smooth centre $C \subset S_{\mathscr{F}(a)}$. By Lemma 4.15, we can assume that $C = V(z, x_k, k \in I) \subset N$, where $I \subset \{1, \ldots, m-1\}$. There is a neighbourhood U of a in N, in which $(x_1, \ldots, x_{m-1}, z)$ form a regular coordinate system. Let N' denote the strict transform of N by σ. Over U, $\sigma\colon N' \to N$ can be identified with $U' = \mathrm{Bl}_C U \to U$. Let $k \in I$ and let $U'_k = U' \backslash V(x_k)'$, where $V(x_k)'$ is the strict transform of $V(x_k) \subset U$. Along the fibre $\sigma^{-1}(a)$ in U'_k, σ is given by a formal coordinate substitution

$$x_k = y_k, \quad x_\ell = y_k y_\ell,\ \ell \in I \cup \{m\}\backslash\{k\}, \quad x_\ell = y_\ell,\ \ell \notin I \cup \{m\}\ .$$

Although y_m here is a formal variable, $y_1, \ldots, y_{m-1}$ are regular functions on U'_k; the fibre $\sigma^{-1}(a)$ is given in U'_k by $y_k = 0$ and $y_\ell = 0$, $\ell \notin I \cup \{m\}$. The above coordinate transformation makes sense as a formal substitution at any $a' \in \sigma^{-1}(a)$ (where y_ℓ, $\ell \in I \cup \{m\}\backslash\{k\}$ need not be zero at a').

Let $a' \in \sigma^{-1}(a) \cap U'_k$. At a', $z' := y_k^{-1}(z \circ \sigma) \sim y_m$ and, for $(f, \mu_f) \in \mathscr{F}(a)$,

$$f'(y) \ :=\ y_k^{-\mu_f}(f \circ \sigma)(y) \ =\ \sum_{0 \le q < \mu_f} c_{f',q}(\widetilde{y}) y_m^q + c_{f',\mu_f}(y) y_m^{\mu_f}\ ,$$

where $c_{f',\mu_f} = c_{f,\mu_f} \circ \sigma$ and $c_{f',q} = y_k^{-(\mu_f - q)} c_{f,q} \circ \widetilde{\sigma}$, $0 \le q < \mu_f$. (We write $\sigma = (\sigma_1, \ldots, \sigma_m)$ and $(\sigma_1, \ldots, \sigma_{m-1}) = \widetilde{\sigma}$.) In particular,

$$\frac{1}{d!} \frac{\partial^{d-1} f'_*}{\partial y_m^{d-1}} \ =\ y_m \left(c_{f'_*,d} + y_k y_m (\cdots) \right)\ .$$

Since y_k vanishes and $c_{f'_*,d}$ is invertible at a', $\partial^{d-1} f'_* / \partial y_m^{d-1} \sim z'$.

Therefore $\mu_{a'}(f') \ge \mu_f$ for all $f \in \mathscr{F}(a)$ (i.e., σ induces a transformation of type (i) at a') if and only if $a' \in V(z')$ and $\mu_{a'} c_{f',q} \ge \mu_f - q$, $0 \le q < \mu_f$, for all $(f, \mu_f) \in \mathscr{F}(a)$. In this case, we have $\mu_{\mathscr{F}(a')} = 1$ and $S_{\mathscr{F}(a')} = \{y \in V(z') :$ $\mu_y(c_{f',q}) \ge \mu_{f'} - q,\ 0 \le q < \mu_{f'}$, for all $(f', \mu_{f'}) = (f', \mu_f) \in \mathscr{F}(a')\}$. Clearly, $\mathscr{E}(a') \cap N'$ comprises $V(y_k)$ and each $V(y_\ell)$, $1 \le \ell \le m-1$ ($\ell \ne k$), such that $V(x_\ell) \in \mathscr{E}(a) \cap N$.

On the other hand, for any $a' \in \sigma^{-1}(a)$, if σ induces a transformation of type (i) at a', then $a' \in U'_k$, for some $k \in I$: To see this, suppose that $a' \in U' \backslash \bigcup_{k \in I} U'_k$. Then σ is given formally at a' by the substitution $x_\ell = y_\ell$, $\ell \notin I$, and $x_\ell = y_\ell y_m$, $\ell \in I$. For each $(f, \mu_f) \in \mathscr{F}(a)$,

$$f'(y) \ :=\ y_m^{-\mu_f}(f \circ \sigma)(y) \ =\ \sum_{0 \le q < \mu_f} c_{f',q}(y) + c_{f',\mu_f}(y)\ ,$$

where $c_{f',\mu_f} = c_{f,\mu_f} \circ \sigma$ and $c_{f',q} = y_m^{-(\mu_f - q)} c_{f,q} \circ \sigma$, $0 \le q < \mu_f$. Since $\mu_{C,a}(c_{f,q}) \ge \mu_f - q \ge 1$, it is clear that $c_{f',q}(0) = 0$, $0 \le q < \mu_f$, for all $(f, \mu_f) \in \mathscr{F}(a)$, and $c_{f'_*,\mu_f}$ is invertible; therefore $\mu_{a'}(f'_*) = 0$.

We have thus established the assertions given by Proposition 4.12 after an admissible blowing-up (transformation of type (i)). The effect of a transformation of type (ii) is trivial, so it remains to consider type (iii).

The effect of an exceptional blowing-up. Let σ be a local blowing-up of M with centre $C = H_0 \cap H_1$, where $H_0, H_1 \in \mathscr{E}(a)$, and let a' denote (the unique point of) $\sigma^{-1}(a) \cap H_1'$ (where H_1' is the strict transform of H_1). We can assume that $H_i \cap N = V(x_{i+1})$, $i = 0, 1$. As above, there is a neighbourhood U of a in N over which σ can be identified with $U' = \mathrm{Bl}_C U \to U$. Then $a' \in U' \backslash V(x_1)'$, and σ is given formally at a' by the substitution $x_1 = y_1$, $x_2 = y_1 y_2$, and $x_\ell = y_\ell$, $\ell > 2$ (where $y(a') = 0$). Of course, $y_1, \ldots, y_{m-1}$ are regular functions at a', and $y_1 = y_{\mathrm{exc}}$.

Since $z \sim x_m$, we have $z' = z \circ \sigma \sim y_m$. For each $(f, \mu_f) \in \mathscr{F}(a)$,

$$f'(y) \ := \ (f \circ \sigma)(y) \ = \sum_{0 \le q < \mu_f} c_{f',q}(\widetilde{y}) y_m^q + c_{f',\mu_f}(y) y_m^{\mu_f} \ ,$$

where $c_{f',q} = c_{f,q}(y_1, y_1 y_2, y_3, \ldots)$, $0 \le q \le \mu_f$. Therefore, $c_{f'_*,d}$ is invertible and, for all $(f, \mu_f) \in \mathscr{F}(a)$, $\mu_{a'}(c_{f',q}) \ge \mu_f - q$, $0 \le q < \mu_f$. Moreover, $\partial^{d-1} f'_* / \partial y_m^{d-1} = (\partial^{d-1} f / \partial x_m^{d-1}) \circ \sigma \sim y_m$. In particular, $\mu_{\mathscr{F}(a')} = 1$ and $S_{\mathscr{F}(a')} = \{y \in V(z') : \mu_y(c_{f',q}) \ge \mu_{f'} - q \ , \ 0 \le q < \mu_{f'}$, for all $(f', \mu_{f'}) = (f', \mu_f) \in \mathscr{F}(a')\}$. (Thus $S_{\mathscr{F}(a')} \subset \sigma^{-1}(S_{\mathscr{F}(a)})$.)

We thus obtain the assertions of 4.12 after an exceptional blowing-up. The proposition follows on repeated application of transformations of type (i), (ii) or (iii). □

Proof of Proposition 4.19. This is essentially a repetition of the proof of Proposition 4.12, but using the expansion of each f in $\mathscr{F}_{r+1}(a)$ with respect to the regular coordinate system $(x_1, \ldots, x_m = z)$ of Construction 4.18. □

Proof of Proposition 4.24. This is a simple consequence of the transformation formulas. Consider h in $\mathscr{H}_{r+1}(a)$, $h = D^d g$ as in Construction 4.23. After a transformation of type (i) (admissible blowing-up σ), the equation

$$y_{\mathrm{exc}}^{-d} h \circ \sigma \ = \ \left(y_{\mathrm{exc}}^{-(1-\nu(a))d} D^d \circ \sigma\right) \cdot \left(y_{\mathrm{exc}}^{-\nu(a)d} g \circ \sigma\right)$$

gives the factorization of $h' = y_{\mathrm{exc}}^{-d} h \circ \sigma$ as $D'^d g'$ (i.e., the analogue for h' of $h = D^d g$) for the following reasons: Except for y_{exc}, any $y_{H'}$, $H' \in \mathscr{E}_{r+1}(a')$, is a common factor of all $g' = y_{\mathrm{exc}}^{-\nu(a)d} g \circ \sigma$ if and only if x_H is a common factor of all g. (H here denotes the element of $\mathscr{E}_{r+1}(a)$ whose strict transform is H'.) On the other hand, for any g such that $\mu_a(g) = \mu_g$, g' is not divisible by y_{exc}.

A similar law for the allowed transformations of type (iii) (exceptional blowings-up) has been implicitly remarked in the second-last paragraph of the proof of 4.11. □

6. inv_X and its key properties

Let M be a manifold and X a closed subspace. Consider an infinitesimal presentation $\left(N_p(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\right)$ of codimension p at $a \in M$. We introduce transforms

X' of X by the three types of morphisms depending on $\big(N_p(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ given by (4.3):

(6.1) (i) If σ is an admissible blowing-up, then X' is the strict transform of X by σ.

(ii), (iii) If σ is either the projection from the product with a line, or an exceptional blowing-up, then X' denotes $\sigma^{-1}(X)$.

Consider a (Zariski-semicontinuous) local invariant $\iota_{X,\cdot}$ of X; for example, $\iota_{X,x} = H_{X,x}$, the Hilbert-Samuel function of X at x, or $\iota_{X,x} = \nu_{X,x}$, the order of X at x. Recall that $S_{\iota_X(a)} := \{x \in |M| : \ \iota_{X,x} \geq \iota_{X,a}\}$. Let $S_\iota(a)$ denote the germ of $S_{\iota_X(a)}$ at a; i.e., the germ at a of the ι_X*-stratum* $\{x : \ \iota_{X,x} = \iota_{X,a}\}$.

Definition 6.2. $\big(N_p(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ *will be called a (***codimension** *p)* **presentation of** $\iota_{X,\cdot}$ **at** a **with respect to** $\mathscr{E}_1(a)$ *if:*

(1) $S_{\mathscr{G}_1(a)} = S_\iota(a)$.

(2) If σ is an admissible blowing-up (4.3) (i) and $a' \in \sigma^{-1}(a)$, then $\iota_{X',a'} = \iota_{X,a}$ if and only if $a' \in N'$ (where N' denotes the strict transform of $N = N_p(a)$) and $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_g} g \circ \sigma) \geq \mu_g$, for all $(g, \mu_g) \in \mathscr{G}_1(a)$.

(3) Consider any finite sequence of transformations of types (i), (ii) and (iii) of $\big(N_p(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ as allowed by $\sim_*$ (Definition 4.10). If $\big(N_p(a'), \mathscr{G}_1(a'), \mathscr{E}_1(a')\big)$ and X' denote the transforms of $\big(N_p(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ and X (respectively) by this sequence, then X', $\iota_{X',a'}$ and $\big(N_p(a'), \mathscr{G}_1(a'), \mathscr{E}_1(a')\big)$ satisfy the analogues of (1), (2) above.

*We define a (***codimension** *p)* **presentation of** $\iota_{X,\cdot}$ **at** *a as a codimension p presentation with respect to $\mathscr{E}_1(a) = \emptyset$.*

Remarks 6.3. (1) Any two presentations of $\iota_{X,\cdot}$ at a with respect to $\mathscr{E}_1(a)$ are equivalent (with respect to $\sim_*$). (2) The equivalence class of a presentation of $\iota_{X,\cdot}$ at a with respect to $\mathscr{E}_1(a)$ depends only on the local isomorphism class of $\big(M, X, \mathscr{E}_1(a)\big)$.

Definition and remark 6.4. We will say $\iota_{X,\cdot}$ admits a *semicoherent presentation* if M can be covered by regular coordinate charts U, such that: (1) $\iota_{X,\cdot}$ has a presentation $\big(N(x), \mathscr{G}_1(x), \mathscr{E}_1(x) = \emptyset\big)$ at each $x \in U$. (codim $N(x)$ may vary with x.) (2) Let $a \in U$. Then there is a Zariski-open neighbourhood V of a in U, together with a regular submanifold N of V (of codimension p, say) and a collection $\mathscr{G}_1 = \{(g, \mu_g)\}$, each defined by data in $\mathscr{O}(U)_V$ (cf. 4.14), such that, for all $x \in S_{\iota_X(a)} \cap V$, $\iota_{X,x} = \iota_{X,a}$ and the germs at x of N and each g give the presentation $\big(N(x), \mathscr{G}_1(x), \mathscr{E}_1(x) = \emptyset\big)$. More generally, we define *semicoherence* of presentations $\big(N(x), \mathscr{G}_1(x), \mathscr{E}_1(x)\big)$ with respect to $\mathscr{E}_1(x)$ by adding the condition that, for all $x \in S_{\iota_X(a)} \cap V$, $\mathscr{E}_1(x) = \{H \in \mathscr{E}_1(a) : \ x \in H\}$.

Suppose that M can be covered by regular coordinate charts U such that, for all $a \in U$, there exists a presentation $\big(N(a), \mathscr{G}_1(a), \mathscr{E}_1(a) = \emptyset\big)$ of $\iota_{X,\cdot}$ at a, defined by data in $\mathscr{O}(U)_a$ (as in 4.14). It follows that $\iota_{X,\cdot}$ admits a semicoherent presentation.

In Chapter III, we will show that the Hilbert-Samuel function admits a semicoherent presentation. However, if X is a hypersurface (i.e., $\mathcal{I}_X$ is principal), this is very simple because $H_{X,\cdot}$ can be replaced by $\nu_{X,\cdot}$ (by Remarks 1.4); the following proposition will allow us to complete the proof of resolution of singularities in the hypersurface case.

Proposition 6.5. *Suppose that X is a hypersurface. Let $a \in X$. If g is a generator of $\mathcal{I}_{X,a}$ (so that g has order $d = \nu_{X,a}$), then $\mathcal{G}_1(a) = \{(g, d)\}$ determines a codimension 0 presentation of $\nu_{X,\cdot}$ at a (such that $\mu_{\mathcal{G}_1(a)} = 1$). Moreover, $\nu_{X,\cdot}$ admits a semicoherent presentation (with codimension 0 and $\mu_{\mathcal{G}_1(\cdot)} = 1$ throughout X).*

Proof. Clearly, $S_{\mathcal{G}_1(a)} = S_\nu(a)$. If σ is an admissible local blowing-up at a ((4.3)(i)) and $a' \in \sigma^{-1}(a)$, then $g' := y_{\mathrm{exc}}^{-d} g \circ \sigma$ generates $\mathcal{I}_{X',a'}$. After a transformation of type (ii) or (iii), $\mathcal{I}_{X',a'}$ is generated by $g' := g \circ \sigma$. If $N_0(a) =$ germ of M at a and $\mathcal{E}_1(a) = \emptyset$, then $\big(N_0(a), \mathcal{G}_1(a), \mathcal{E}_1(a)\big)$ is an infinitesimal presentation of codimension 0 satisfying the hypotheses of Proposition 4.12. The first assertion follows from 4.12 and 4.13. Since $\mathcal{I}_X$ is of finite type, it is clear that we can choose a codimension 0 semicoherent presentation of $\nu_{X,\cdot}$ (with the function g of 6.4 regular on each coordinate chart U). $\square$

Suppose that X is a hypersurface. We can now use the inductive construction of Sect. 4, beginning with a (semicoherent) presentation of $\nu_{X,\cdot}$, to define $\mathrm{inv}_X(a)$ and prove Theorem 1.14. Exactly the same arguments will apply to the general case once we obtain a presentation of the Hilbert-Samuel function $H_{X,\cdot}$.

We begin with a general proposition that will be used to establish "semicontinuity" of the exceptional sets $E^r(a)$. (See Sect. 1.) Consider a sequence of transformations

$$\begin{array}{ccccccccc} M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 = M \\ E_{j+1} & & E_j & & & & E_1 & & E_0 = \emptyset \end{array}$$

where, for each j, σ_{j+1} is a local blowing-up with smooth centre C_j such that C_j and E_j simultaneously have only normal crossings, and E_{j+1} is the collection of smooth hypersurfaces $\{\sigma_{j+1}^{-1}(C_j)$ and H', for all $H \in E_j\}$ (where H' denotes the strict transform of H). If $a \in M_j$, we set $E(a) = \{H \in E_j : a \in H\}$. Let ι denote a function with values in a partially-ordered set, defined on each M_j, with the following properties: ι is constant on each C_j, ι is Zariski-semicontinuous on each M_j, and ι is infinitesimally upper-semicontinuous (i.e., if $a \in M_j$, then $\iota(a) \le \iota\big(\sigma_j(a)\big)$). If $a \in M_j$, let i denote the smallest index k such that $\iota(a) = \iota(a_k)$ (where $a_k := (\sigma_{k+1} \circ \cdots \circ \sigma_j)(a)$), and set $E_\iota(a) = \{H \in E(a) : H$ is the strict transform of some element of $E(a_i)\}$.

Proposition 6.6. *Let $a \in M_j$, for some j. Then there is a Zariski-open neighbourhood U of a in M_j such that, for all $x \in S_{\iota(a)} \cap U$, $E_\iota(x) = E(x) \cap E_\iota(a)$.*

Proof. By induction, we can assume the result in M_k, $k < j$. Then there is a Zariski-open neighbourhood U of a such that, if $x \in U$, then: (1) $\iota(x) \le \iota(a)$; (2)

$\iota(x_{j-1}) \le \iota(a_{j-1})$; (3) $E(x) \subset E(a)$; (4) $E_\iota(x_{j-1}) = E(x_{j-1}) \cap E_\iota(a_{j-1})$ whenever $x_{j-1} \in S_{\iota(a_{j-1})}$; (5) $x_k \notin C_k$ if $a_k \notin C_k$, for all $k < j$. We consider 3 cases:

(a) $\iota(a) = \iota(a_{j-1})$. Since $\iota(a) = \iota(x) \le \iota(x_{j-1}) \le \iota(a_{j-1})$, all terms are equal. Thus $E_\iota(a) = E(a) \cap \{H' : H \in E_\iota(a_{j-1})\}$ and $E_\iota(x) = E(x) \cap \{H' : H \in E_\iota(x_{j-1})\}$. By (4), $E_\iota(x) = E(x) \cap E_\iota(a)$.

(b) $\iota(a) < \iota(a_{j-1})$ and $\iota(x) < \iota(x_{j-1})$. Then, by definition, $E_\iota(a) = E(a)$ and $E_\iota(x) = E(x)$, so the result follows trivially from (3).

(c) $\iota(a) < \iota(a_{j-1})$ but $\iota(x) = \iota(x_{j-1})$. Then $\iota(x_{j-1}) < \iota(a_{j-1})$, so $x_{j-1} \notin C_{j-1}$ and σ_j induces an isomorphism between neighbourhoods of x and x_{j-1} (taking $E(x)$ to $E(x_{j-1})$ and $E_\iota(x)$ to $E_\iota(x_{j-1})$). We have $E_\iota(a) = E(a)$, so we have to prove $E_\iota(x) = E(x)$: Let i be the least k such that $\iota(x_k) = \iota(x)$. Then $i < j$ and $\iota(x) = \iota(x_{j-1}) = \cdots = \iota(x_i)$. Thus, for all $k = i, \ldots, j-1$, $\iota(x_k) < \iota(a_k)$. It follows from (5) that $x_k \notin C_k$ and $E(x) \cong E(x_k)$, $E_\iota(x) \cong E_\iota(x_k)$. Then $E_\iota(x_i) = E(x_i)$, so $E_\iota(x) = E(x)$. $\square$

Definition of $\mathbf{inv_X}$. We consider a sequence of transformations

$$
(6.7)\qquad
\begin{array}{ccccccccc}
\longrightarrow & M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 = M \\
 & X_{j+1} & & X_j & & & & X_1 & & X_0 = X \\
 & E_{j+1} & & E_j & & & & E_1 & & E_0 = E
\end{array}
$$

where, for each j,

(1) σ_{j+1}: $M_{j+1} = \mathrm{Bl}_{C_j} W_j \to W_j \hookrightarrow M_j$ is a local blowing-up with smooth centre $C_j \hookrightarrow W_j$, and C_j, E_j simultaneously have only normal crossings.

(2) X_{j+1} is the strict transform X_j' of X_j by σ_{j+1}.

(3) $E_{j+1} = \{H' : H \in E_j\} \cup \{\sigma_{j+1}^{-1}(C_j)\}$, where H' denotes the strict transform of H. (By (1), E_{j+1} has only normal crossings.)

If $a \in M_j$, set $E(a) = \{H \in E_j : a \in H\}$. Write $\sigma_{ij} = \sigma_{i+1} \circ \cdots \circ \sigma_j$, $i = 0, \ldots, j-1$, and $\sigma_{jj} =$ identity. If $a \in M_j$, set $a_i = \sigma_{ij}(a)$, $i = 0, \ldots, j$.

For all $a \in M_j$, $j \ge 0$, set $\mathrm{inv}_{1/2}(a) = H_{X_j,a}$ (or $\mathrm{inv}_{1/2}(a) = \nu_1(a)$, where $\nu_1(a) := \nu_{X_j,a}$, if X is a hypersurface). Then $\mathrm{inv}_{1/2}$ is Zariski-semicontinuous on each M_j.

Assume now that each centre of blowing-up C_j is $1/2$-*admissible*; i.e., $\mathrm{inv}_{1/2}$ is (locally) constant on C_j. If X is a hypersurface, then $\mathrm{inv}_{1/2}$ is infinitesimally upper-semicontinuous (i.e., $\mathrm{inv}_{1/2}(a') \le \mathrm{inv}_{1/2}(a)$ for all $a \in M_j$ and $a' \in \sigma_{j+1}^{-1}(a)$, $j \ge 0$) by 5.1. In general, $\mathrm{inv}_{1/2}$ is infinitesimally upper-semicontinuous by Theorem 7.20.

Definitions 6.8. *Suppose $a \in M_j$. Let i be the smallest k such that* $\mathrm{inv}_{1/2}(a) = \mathrm{inv}_{1/2}(a_k)$, *and set $E^1(a) = \{H \in E(a)$: H is the strict transform of some element of $E(a_i)\}$. Put $\mathscr{E}_1(a) = E(a) \setminus E^1(a)$. Set $s_1(a) = \# E^1(a)$, and* $\mathrm{inv}_1(a) = \big(\mathrm{inv}_{1/2}(a), s_1(a)\big)$.

Clearly, $\mathrm{inv}_1(a)$ is a local invariant of the triple $\big(M_j, X_j, E^1(a)\big)$. It follows from Proposition 6.6 that inv_1 is Zariski-semicontinuous on M_j, for all j. It is also clear that inv_1 is infinitesimally upper-semicontinuous.

We now assume that X is a hypersurface, so that we can use Proposition 6.5 above. But all of the following arguments hold in general, once we construct a semicoherent presentation of the Hilbert-Samuel function.

Proposition 6.9. *For all $a \in M_j$, $j \geq 0$, there is a (codimension zero) presentation $\big(N_0(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ of $\nu_1 = \nu_{X_j,\cdot}$ at a with respect to $\mathscr{E}_1(a)$ satisfying the hypotheses of 4.12. Such presentations can be chosen for all $a \in M_j$ in a semicoherent way.*

Proof . Let $a \in M_j$, $j \geq 0$. Let i denote the smallest index k such that $\nu_1(a) = \nu_1(a_k)$. Then $\mathscr{E}_1(a_i) = \emptyset$. By Proposition 6.5, there is a codimension 0 presentation $\big(N_0(a_i), \mathscr{G}_1(a_i), \mathscr{E}_1(a_i) = \emptyset\big)$ of ν_1 at a_i, where $\mu_{\mathscr{G}_1(a_i)} = 1$. Inductively, for each $k = i, \dots, j-1$, σ_{k+1} induces a transformation of type (i), $\big(N_0(a_k), \mathscr{G}_1(a_k), \mathscr{E}_1(a_k)\big) \mapsto \big(N_0(a_{k+1}), \mathscr{G}_1(a_{k+1}), \mathscr{E}_1(a_{k+1})\big)$. Then $\big(N_0(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ is a presentation of ν_1 at a with respect to $\mathscr{E}_1(a)$, which satisfies the hypothesis of Proposition 4.12, as in Example 4.16. The second assertion follows from Propositions 6.5 and 6.6, and Remarks 4.14. □

Let $a \in M_j$, $j \geq 0$. Let $\big(N_0(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ be as in Proposition 6.9, and define

$$\mathscr{F}_1(a) \ := \ \mathscr{G}_1(a) \cup \big(E^1(a), 1\big) \ ,$$

where $\big(E^1(a), 1\big) := \{(\ell_H, 1) : \ H \in E^1(a)\}$ and $\ell_H \in \mathscr{O}_{M,a}$ generates $\mathscr{I}_{H,a}$. Clearly, $\big(N_0(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$ satisfies the hypotheses of 4.12 (as in 4.16), and its equivalence class (always with respect to $\sim_*$) depends only on $E^1(a)$ and that of $\big(N_0(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$; thus only on the local isomorphism class of $\big(M, X, E, E^1(a)\big)$. It follows from 6.6 and 6.9 that $\big(N_0(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$ can be chosen for all $a \in M_j$ in a semicoherent way.

Moreover, $\big(N_0(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$ is a (*codimension* 0) *presentation of* inv_1 *at* a *with respect to* $\mathscr{E}_1(a)$ in the sense that:

(6.10) (1) $S_{\mathscr{F}_1(a)} = S_{\mathrm{inv}_1(a)}$.

(2) If σ is a local blowing-up at a where centre C is 1-*admissible* (i.e., inv_1 is constant on C) and $a' \in \sigma^{-1}(a)$, then $\mathrm{inv}_1(a') = \mathrm{inv}_1(a)$ if and only if $\mu_{a'}(y_{\mathrm{exc}}^{-\mu_f} f \circ \sigma) \geq \mu_f$, for all $(f, \mu_f) \in \mathscr{F}_1(a)$.

(3) The analogues of (1), (2) hold after any sequence of transformations of type (i) (1-admissible local blowings-up).

Remark 6.11. It is possible to define inv_1 after transformations of types (ii) and (iii) as well, so that "presentation of inv_1" can be defined in complete analogy with Definition 6.2. But we do not need this; transformations of types (ii) and (iii) are used only to establish invariance of $\nu_r(a)$, $r > 1$, by test blowings-up (Propositions 4.8 and 4.11).

In summary: Given a sequence of local blowings-up (6.7) with centres which are $1/2$-admissible, we take $\mathscr{E}_1(a) = E(a) \backslash E^1(a)$, $a \in M_j$, and introduce a (semicoherent) presentation $\big(N_0(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ of $\mathrm{inv}_{1/2}$ at a with respect to $\mathscr{E}_1(a)$, of codimension 0 (in the hypersurface case). Setting $\mathrm{inv}_1 = (\mathrm{inv}_{1/2}, s_1)$, where

$s_1(a) = \#E^1(a)$, and adjoining $\big(E^1(a), 1\big)$ to $\mathscr{G}_1(a)$, we obtain a (semicoherent) presentation $\big(N_0(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$ of inv_1 at a with respect to $\mathscr{E}_1(a)$.

By 4.18, 4.19 (and 4.20), there is an equivalent (semicoherent) codimension 1 presentation $\big(N_1(a), \mathscr{H}_1(a), \mathscr{E}_1(a)\big)$ of inv_1 at a with respect to $\mathscr{E}_1(a)$.

Definitions and remarks 6.12. We define $\mu_2(a) = \mu_{\mathscr{H}_1(a)}$ as in (4.21). It follows from Proposition 4.8 that $\mu_2(a)$ is a local invariant of $\big(M, X, E, E^1(a)\big)$; $1 \le \mu_2(a) \le \infty$ and $\mu_2(a) = \infty$ if and only if $\mathscr{H}_1(a) = 0$. If $\mu_2(a) < \infty$, then (as in (4.22)) we set

$$\begin{aligned} \mu_{2,H}(a) &= \mu_{\mathscr{H}_1(a),H}\ , \qquad H \in \mathscr{E}_1(a)\ , \\ \nu_2(a) &= \mu_2(a) - \sum_{H \in \mathscr{E}_1(a)} \mu_{2,H}(a)\ . \end{aligned}$$

Thus $\nu_2(a) \ge 0$. Also set $\nu_2(a) = \infty$ if $\mu_2(a) = \infty$. Put $\mathrm{inv}_{1\frac{1}{2}}(a) = \big(\mathrm{inv}_1(a); \nu_2(a)\big)$. It follows from 4.11 that $\mathrm{inv}_{1\frac{1}{2}}(a)$ is a local invariant of $\big(M, X, E, E^1(a)\big)$.

Proposition 6.13. $\mathrm{inv}_{1\frac{1}{2}}$ *is Zariski-semicontinuous on each* M_j*. If* C_j *is* $1\frac{1}{2}$*-admissible (i.e.,* $\mathrm{inv}_{1\frac{1}{2}}$ *is (locally) constant on* C_j*), then* $\mathrm{inv}_{1\frac{1}{2}}(a') \le \mathrm{inv}_{1\frac{1}{2}}(a)$ *for all* $a' \in \sigma^{-1}(a)$*.*

Remark 6.14. In the case of analytic spaces over $\underline{k}$, where $\underline{k}$ is not algebraically closed, the argument following actually shows that $\mathrm{inv}_{1\frac{1}{2}}$ (or, more generally, $\mathrm{inv}_{r+\frac{1}{2}}$, $r \ge 1$, and thus inv_X) is "Zariski-semicontinuous" in the weaker sense of the paragraph following Definition 3.11. This suffices for all of our results, except for canonical desingularization in the noncompact case. In fact, though, Zariski-semicontinuity (in the sense of Definition 3.11) follows because $\mathrm{inv}_{r+\frac{1}{2}}$ is invariant under any finite field extension, and a germ of an analytic function which vanishes on $S_{\mathrm{inv}_{r+\frac{1}{2}}(a)}$ at some point will vanish also on $S_{\mathrm{inv}_{r+\frac{1}{2}}(a)}$ when defined over a finite extension of $\underline{k}$.

Proof of Proposition 6.13. This follows from Constructions 4.18 and 4.23, and from semicoherence of $\big(N_0(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$. Recall that $\mu_h = d$ for all $(h, \mu_h) \in \mathscr{H}_1(a)$ (in the notation of 4.23). On $S_{\mathrm{inv}_1}(a)$, $\nu_2(x) = \frac{1}{d}\min_{\mathscr{H}_1(a)} \mu_x(D_2^{-d}h)$. The first assertion follows, therefore, from semicontinuity of multiplicity. The second assertion follows from Lemma 5.1 because, if C_j is $1\frac{1}{2}$-admissible, $\mathrm{inv}_1(a') = \mathrm{inv}_1(a)$ where $a' \in \sigma_{j+1}^{-1}(a)$, and g denotes an element of minimal order among the h/D_2^d, then g transforms by σ_{j+1} at a' according to the law in Lemma 5.1 (cf. proof of Proposition 4.24). $\square$

If $\nu_2(a) = 0$ or ∞, set $\mathrm{inv}_X(a) = \mathrm{inv}_{1\frac{1}{2}}(a)$. Suppose $0 < \nu_2(a) < \infty$. Construction 4.23 provides a (codimension 1) infinitesimal presentation $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_1(a)\big)$ such that $\mu_{\mathscr{G}_2(a)} = 1$, whose equivalence class (with respect to $\sim_*$) depends only on that of $\big(N_0(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$ (by 4.24). Moreover, $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_1(a)\big)$ is a presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a with respect to $\mathscr{E}_1(a)$, in analogy with (6.10). Using

Remark 4.25, we see that $\mathrm{inv}_{1\frac{1}{2}}$ admits a semicoherent (codimension 1) presentation on each M_j. (If $a \in M_j$ and $\nu_2(a) = 0$ or ∞, then $S_{\mathrm{inv}_{1\frac{1}{2}}}(a) = S_{\mathrm{inv}_1}(a)$ and $\big(N_1(a), \mathscr{H}_1(a), \mathscr{E}_1(a)\big)$, as defined above, is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a with respect to $\mathscr{E}_1(a)$.)

Now let us assume that the centres C_j of the σ_{j+1} in (6.7) are all $1\frac{1}{2}$-admissible.

Definitions 6.15. *Suppose $a \in M_j$ and $0 < \nu_2(a) < \infty$. Let i denote the smallest k such that $\mathrm{inv}_{1\frac{1}{2}}(a) = \mathrm{inv}_{1\frac{1}{2}}(a_k)$, and set $E^2(a) = \{H \in \mathscr{E}_1(a) : H$ is the strict transform of some element of $\mathscr{E}_1(a_i)\}$. Put $\mathscr{E}_2(a) = \mathscr{E}_1(a)\backslash E^2(a)$. We set $s_2(a) = \#E^2(a)$, and $\mathrm{inv}_2(a) = \big(\mathrm{inv}_{1\frac{1}{2}}(a), s_2(a)\big)$. (If $\nu_2(a) = 0$ or ∞, we set $\mathrm{inv}_2(a) = \mathrm{inv}_X(a)$.)*

Clearly, $\mathrm{inv}_2(a)$ is a local invariant of $\big(M_j, X_j, E(a), E^1(a), E^2(a)\big)$. It follows from Proposition 6.6 that inv_2 is Zariski-semicontinuous on M_j, for all j. It is also clear that inv_2 is infinitesimally upper-semicontinuous.

When $0 < \nu_2(a) < \infty$, we continue inductively: Let $a \in M_j$, $j = 0, 1, \ldots$, and let $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_1(a)\big)$ be a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a with respect to $\mathscr{E}_1(a)$. Then $\big(N_1(a), \mathscr{G}_2(a), \mathscr{E}_2(a)\big)$ is a codimension 1 presentation of $\mathrm{inv}_{1\frac{1}{2}}$ at a with respect to $\mathscr{E}_2(a)$ *that satisfies the hypothesis of Proposition 4.12* (as in 4.16; cf. proof of 6.9). Define $\mathscr{F}_2(a) := \mathscr{G}_2(a) \cup \big(E^2(a), 1\big)$. Clearly, $\big(N_1(a), \mathscr{F}_2(a), \mathscr{E}_2(a)\big)$ satisfies the hypotheses of 4.12, and its equivalence class depends only on the local isomorphism class of $\big(M_j, X_j, E(a), E^1(a), E^2(a)\big)$. Moreover, $\big(N_1(a), \mathscr{F}_2(a), \mathscr{E}_2(a)\big)$ is a codimension 1 presentation of inv_2 at a with respect to $\mathscr{E}_2(a)$. Then inv_2 admits a semicoherent presentation on each M_j (using $\big(N_1(\cdot), \mathscr{F}_2(\cdot), \mathscr{E}_2(\cdot)\big)$ on the set where $0 < \nu_2(\cdot) < \infty$).

We thus continue inductively, first to define $\nu_{r+1}(a)$, and then $s_{r+1}(a)$ after assuming that all centres C_j in (6.7) are $(r + \frac{1}{2})$-admissible. In general, of course, the semicoherent presentation that we construct for $\mathrm{inv}_{r+\frac{1}{2}}$ or inv_{r+1} will have codimension that varies according to the stratum. Eventually, we reach $t \le n = \dim_a M_j$ such that $0 < \nu_r(a) < \infty$, $r \le t$, and $\nu_{t+1}(a) = 0$ or ∞. Then we define $\mathrm{inv}_X(a) = \big(\mathrm{inv}_t(a); \nu_{t+1}(a)\big)$. In this case, already $S_{\mathrm{inv}_t}(a) = S_{\mathrm{inv}_X}(a)$. Our construction provides a codimension t presentation $\big(N_t(a), \mathscr{H}_t(a), \mathscr{E}_t(a)\big)$ of inv_t (or of inv_X) at a with respect to $\mathscr{E}_t(a)$.

Remark 6.16. Suppose that $\mathrm{inv}_{1/2} = \nu_1$ admits a codimension p presentation $\big(N_p(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ at $a \in M_j$, where $p \ge 1$. Then $\mathrm{inv}_p(a) = (\nu_1(a), s_1(a); 1, 0; \ldots;\ 1, 0)$ (i.e., $(1, 0)$ is listed $p - 1$ times). Moreover, if $\big(N_1(a), \mathscr{C}_1(a), \mathscr{E}_1(a)\big)$ is a codimension 1 presentation of $\mathrm{inv}_{1/2}$ at a, then our construction provides an (equivalent) codimension p presentation $\big(N_p'(a), \mathscr{G}_1'(a), \mathscr{E}_1(a)\big)$, where $N_p'(a) \subset N_1(a)$.

Proof of Theorem 1.14 on the key properties of $\mathbf{inv_X}$. We have already seen that the semicontinuity property (1) of Theorem 1.14 follows from our construction. To prove (2): The stabilization property of $\mathrm{inv}_{1/2} = \nu_1$ is a consequence of infinitesimal semicontinuity because $\nu_1(a) \in \mathbb{N}$. The assertion for inv_X follows from infinitesimal semicontinuity of inv_X because, although $\nu_{r+1}(a)$, for each $r > 0$, is perhaps merely rational, our construction above shows that $e_r!\nu_{r+1}(a) \in \mathbb{N}$, where $e_1 = \nu_1(a)$ and $e_{r+1} = \max\{e_r!, e_r!\nu_{r+1}(a)\}$, $r > 0$. It remains to get properties (3) and (4).

Case (a). $\nu_{t+1}(a) = \infty$. Then (3) and (4) are both trivial because $S_{\mathrm{inv}_t}(a) = N_t(a)$ and, if σ is the local blowing-up with centre $N = N_t(a)$, then the strict transform $N' = \emptyset$, so that $\mathrm{inv}_t(a') < \mathrm{inv}_t(a)$, for all $a' \in \sigma^{-1}(a)$ (cf. 6.2(2)).

Case (b). $\nu_{t+1}(a) = 0$. We use the notation of (4.22) and Construction 4.23. We have $h = D_{t+1}^d$, for some $(h, \mu_h = d) \in \mathscr{H}_t(a)$, and

$$S_{\mathrm{inv}_t}(a) \;=\; \{x = (x_1, \dots, x_{n-t}) \in N_t(a) :\; \mu_x(D_{t+1}) \geq 1\}\ .$$

(Recall that $D_{t+1}(x)$ is a monomial $x_1^{\Omega_1}\cdots x_{n-t}^{\Omega_{n-t}}$ with rational exponents; if $\Omega_\ell \neq 0$, then $x_\ell = x_H$ for some $H \in \mathscr{E}_t(a)$, and $\Omega_\ell = \mu_{t+1,H}(a)$. Thus $\mu_x(D_{t+1})$ makes sense as a rational number.) Therefore, $S_{\mathrm{inv}_t}(a)$ is a union of smooth components $\bigcup_I Z_I$, where $Z_I = \{x \in N_t(a) :\; x_\ell = 0,\; \ell \in I\}$ and the union is over the minimal subsets I of $\{1, \dots, n-t\}$ such that $\sum_{\ell\in I} \Omega_\ell \geq 1$; equivalently, over the subsets I such that

$$0 \;\leq\; \sum_{m\in I} \Omega_m - 1 \;<\; \Omega_\ell\ , \qquad\qquad \text{for all } \ell \in I\ .$$

In particular, (3) holds.

We prove (4) using $\mu_X(a) = \mu_{t+1}(a)$. Consider a local blowing-up $\sigma\colon W' \to W \hookrightarrow M_j$ with centre Z_I, for some I as above. (W is a neighbourhood of $a = 0$ in which $(x_1, \dots, x_{n-1})$ extend to regular coordinates for $N = N_t(a)$.) Suppose $a' \in \sigma^{-1}(a)$ and $\mathrm{inv}_t(a') = \mathrm{inv}_t(a)$. Then $a' \in N'$, where N' is the strict transform of N; N' is a union of regular coordinate charts $\bigcup_{\ell\in I} U'_\ell$ such that $\sigma \mid U'_\ell$ is given by $x_\ell = y_\ell$, $x_m = y_\ell y_m$ if $m \in I\backslash\{\ell\}$, and $x_m = y_m$ if $m \notin I$. Consider $h = D_{t+1}^d \in \mathscr{H}_t(a)$. Suppose $a' \in U'_\ell$. Then $h' \in \mathscr{H}_t(a')$, where $h' \;:=\; y_\ell^{-d}(D_{t+1}^d)\circ\sigma \;=\; (y_1^{\Omega'_1}\cdots y_{n-t}^{\Omega'_{n-t}})^d$, and

$$\Omega'_m \;=\; \Omega_m\ , \quad m \neq \ell\ , \qquad\qquad \Omega'_\ell \;=\; \sum_{m\in I} \Omega_m - 1 \;<\; \Omega_\ell\ .$$

Therefore, $1 \leq \mu_{t+1}(a') \leq \sum_{m=1}^{n-t} \Omega'_m < \sum_{m=1}^{n-t} \Omega_m = \mu_{t+1}(a)$, as required. □

Remark 6.17. Suppose $a \in M_j$ and $\mathrm{inv}_X(a) = \big(\mathrm{inv}_t(a); \nu_{t+1}(a)\big)$, where $\nu_{t+1}(a) = 0$ or ∞ as above. Consider the extended invariant $\mathrm{inv}_X^{\mathrm{e}}(a) = \big(\mathrm{inv}_X(a); J(a)\big)$, with $J(a)$ defined as in Remarks 1.16. Note that $\bigcup_{r\leq t} E^r(a) \subset J(a)$. Set $J_t(a) := J(a)\backslash \bigcup_{r\leq t} E^r(a)$. Let $S_X^{\mathrm{e}}(a) := S_{\mathrm{inv}_X^{\mathrm{e}}}(a)$. If $\nu_{t+1}(a) = \infty$, then $\operatorname{codim} S_X^{\mathrm{e}}(a) = t$, and if $\nu_{t+1}(a) = 0$, then $\operatorname{codim} S_X^{\mathrm{e}} = t + \#J_t(a)$.

Chapter III. Presentation of the Hilbert-Samuel function; desingularization in the general case

In Sect. 9, we construct a semicoherent presentation of the Hilbert-Samuel function (Definitions 6.2, 6.4), so that resolution of singularities follows as in Ch.II (cf. Remarks 9.15). Our presentation at a point is formally equivalent to a "formal presentation" in codimension zero. The standard basis of the ideal $\widehat{\mathcal{I}}_{X,a} \subset \widehat{\mathcal{O}}_{M,a} \cong \underline{k}[[x]]$ with respect to coordinates $x = (x_1, \ldots, x_n)$ provides a formal presentation of $H_{X,\cdot}$ at a (cf. [BM4, Theorem 7.3]), but does not, in general, correspond to a regular presentation (i.e., a presentation in the sense of 6.2, which is given by regular functions as in 4.1; cf. Remark 1.19). Our regular presentation is determined by the coefficients (of an expansion with respect to "essential variables") of a system of generators of $\widehat{\mathcal{I}}_{X,a}$ satisfying properties which isolate the essential features of a standard basis that are preserved by transformations of types (i), (ii) and (iii) ((4.4) above). Sect. 7 is purely formal: we introduce these properties and prove they determine a presentation of the Hilbert-Samuel function associated to an ideal in a ring of formal power series. Semicoherence of the regular presentation of Sect. 9 depends on showing that the formal properties are open (in the Zariski topology) on the Hilbert-Samuel stratum $S_H(a) = \{x : \ H_{X,x} = H_{X,a}\}$. A combinatorial stabilization theorem for the diagram of initial exponents (Sect. 8) plays an important part.

7. The formal presentation

Let $\underline{k}[[X]] = \underline{k}[[X_1, \ldots, X_n]]$. If $G = (G_1, \ldots, G_q)$, where each $G_j \in \underline{k}[[X]]$, we write (G) or $(G_1, \ldots, G_q)$ for the ideal generated by the G_j. Let $\mathfrak{N} \in \mathcal{D}(n)$ (cf. 3.18 ff.), and let $H_{\mathfrak{N}} \colon \mathbb{N} \to \mathbb{N}$ denote the function $H_{\mathfrak{N}}(k) = \#\{\alpha \in \mathbb{N}^n \backslash \mathfrak{N} : \ |\alpha| \leq k\}$.

(7.1) Structure of the diagram. Let α^i, $i = 1, \ldots, s$, be the vertices of $\mathfrak{N}$ in ascending order. (We totally order $\mathbb{N}^n$ using the lexicographic ordering of $(|\alpha|, \alpha_1, \ldots, \alpha_n)$, $\alpha \in \mathbb{N}^n$.) For each $k \in \mathbb{N}$, set $s(k) = \max\{i : \ |\alpha^i| \leq k\}$ and put $\mathfrak{N}(k) = \bigcup_{i=1}^{s(k)} (\alpha^i + \mathbb{N}^n)$. We group the vertices α^i into blocks of given order $|\alpha^i|$; say that $k_1 < k_2 < \cdots < k_p$ are the orders of the blocks. Let $s_\ell = s(k_\ell)$, $\ell = 1, \ldots, p$. Then $s_1 < s_2 < \cdots < s_p = s$ and $\alpha^1, \ldots, \alpha^{s_\ell}$ are the vertices of $\mathfrak{N}$ with $|\alpha^i| \leq k_\ell$, for each $\ell = 1, \ldots, p$.

By a possible permutation of the variables, we can assume that the last r indeterminates $(X_{n-r+1}, \ldots, X_n)$ are precisely the "essential variables" of the monomials X^{α^i}; i.e., the variables occurring to positive power in some X^{α^i}. ("Essential variables" is used here in a weaker sense than in [BM4, Sect. 6].) Thus $\mathfrak{N} = \mathbb{N}^{n-r} \times \mathfrak{N}^*$, where $\mathfrak{N}^* \in \mathcal{D}(r)$ and r is as small as possible for any permutation of the variables. (r is not determined by the Hilbert-Samuel function $H_{\mathfrak{N}}$; cf. 9.15(1).) Obviously, each $\alpha^i \in \{0\} \times \mathfrak{N}^*$. Write $X = (W, Z) = (W_1, \ldots, W_{n-r}, Z_1, \ldots, Z_r)$. We can assume in the same way that we have $1 \leq$

$r_1 \le r_2 \le \cdots \le r_p = r$ so that, for each $\ell = 1, \ldots, p$, the last r_ℓ variables $Z^\ell = (Z_{r-r_\ell+1}, \ldots, Z_r)$ are the essential variables of the monomials X^{α_i}, $1 \le i \le s_\ell$ (corresponding to the first ℓ blocks of vertices). It follows that, for all $j = 1, \ldots, r$, there exists $i(j)$, $1 \le i(j) \le s$, such that if $j > r - r_\ell$, then $i(j) \le s_\ell$ and $\alpha^{i(j)} = \beta^j + \big(0, (j)\big) \in \mathbb{N}^{n-r} \times \mathbb{N}^r$, where $\beta^j \in \{0\} \times \mathbb{N}^{r_\ell} \subset \mathbb{N}^{n-r_\ell} \times \mathbb{N}^{r_\ell}$ (and $(j) \in \mathbb{N}^r$ is the multiindex with 1 in the j'th place and 0 elsewhere).

Each $\mathfrak{N}(k_\ell)$ has the form $\mathfrak{N}(k_\ell) = \mathbb{N}^{n-r_\ell} \times \mathfrak{N}^\ell$, where $\mathfrak{N}^\ell = \mathfrak{N}(k_\ell)^* \subset \mathbb{N}^{r_\ell}$, and each $\Box_i = \mathbb{N}^{n-r_\ell} \times \Box_i^\ell$, where $\Box_i^\ell \subset \mathbb{N}^{r_\ell}$. (We write $\mathfrak{N} = \bigcup(\alpha^i + \Box_i)$ as in Sect. 3.)

The formal properties. Let I denote an ideal in $\underline{k}[[X]]$, and let $f_i(X) = f_i(W, Z) \in I$, $i = 1, \ldots, s$. Let $\mathfrak{N} \in \mathscr{D}(n)$. Fix $K \in \mathbb{N}$, $K \ge \max|\alpha^i| - 1$. We consider the following five properties (using the notation of (7.1)):

(7.2) (1) $\mu(f_i) = |\alpha^i|$, $i = 1, \ldots, s$. ($\mu(f)$ denotes the order of $f \in \underline{k}[[X]]$.)

(2) *Division property of the initial forms* $(\mathrm{in} f_i)\,(X) = \sum_{|\alpha|=d_i} D^\alpha f_i(0) X^\alpha/\alpha!$, where $d_i = |\alpha^i|$. For each $k \in \mathbb{N}$: (2_k) If $f(X)$ is a homogeneous polynomial, say of degree d, then there exist unique homogeneous polynomials $Q_i(f)$ of degrees $d - d_i$, $i = 0, \ldots, s(k)$ (where $d_0 = 0$ and $Q_i(f) = 0$ if $d < d_i$) such that

$$f = \sum_{i=1}^{s(k)} Q_i(f) \cdot \mathrm{in} f_i + Q_0(f)\,,$$

$\mathrm{supp}\, Q_i(f) \subset \Box_i$, $i = 1, \ldots, s(k)$, and $\mathrm{supp}\, Q_0(f) \cap \mathfrak{N}(k) = \emptyset$. Of course, (2_k) is equivalent to "$(2_{k,d})$ for all $d \in \mathbb{N}$", where $(2_{k,d})$ is the condition that

$$\begin{array}{ll} X^\beta (\mathrm{in} f_i)(X), & \beta \in \Box_i\,,\ |\beta| = d - d_i\,,\quad i = 1, \ldots, s(k), \\ X^\gamma, & \gamma \notin \mathfrak{N}(k),\ |\gamma| = d, \end{array}$$

span the $\underline{k}$-vector space $(X)^d/(X)^{d+1}$ of homogeneous polynomials of degree d.

(3) For all $f \in I$, there exist $q_i(f) \in \underline{k}[[X]]$, $i = 1, \ldots, s$, such that $f = \sum_{i=1}^{s} q_i(f) \cdot f_i$ and $\mathrm{supp}\, q_i(f) \subset \Box_i$, for each i.

(4) For all $j = 1, \ldots, r$, let $g_j(X) = D^{\beta_j} f_{i(j)}(X)$. Then for each $\ell = 1, \ldots, p$: (4_ℓ) $g_j(X) \in (Z^\ell)$ if $r - r_\ell < j \le r$, and $\det(\partial g^\ell/\partial Z^\ell)(0) \neq 0$, where $\partial g^\ell/\partial Z^\ell$ denotes the Jacobian matrix of $g^\ell := (g_{r-r_\ell+1}, \ldots, g_r)$ with respect to $Z^\ell = (Z_{r-r_\ell+1}, \ldots, Z_r)$, and (Z^ℓ) denotes the ideal generated by $Z_{r-r_\ell+1}, \ldots, Z_r$.

(5) For each $\ell = 1, \ldots, p$: (5_ℓ) If $i > s_\ell$, then $D^\beta_{Z^\ell} f_i \in (Z^\ell)$, for all $\beta \in \mathfrak{N}^\ell \subset \mathbb{N}^{r_\ell}$, $|\beta| \le K$. ($D^\beta_{Z^\ell}$ denotes the formal partial derivative of order β with respect ot Z^ℓ.)

We begin with some elementary remarks on properties (1)–(3) of (7.2) and their relationship with the Hilbert-Samuel function of $\underline{k}[[X]]/I$. Let $f_i(X) \in I$, $i = 1, \ldots, s$. The following can be proved by Euclidean division (cf. Theorem 3.17).

Lemma 7.3. *(Formal division algorithm). Let $k \in \mathbb{N}$. Assume properties (1) and (2_k). Then for all $f \in \underline{k}[[X]]$, there are unique $q_i(f) \in \underline{k}[[X]]$, $i = 0, \dots, s(k)$, such that*

$$f = \sum_{i=1}^{s(k)} q_i(f) f_i + q_0(f) ,$$

$\operatorname{supp} q_i(f) \subset \Box_i$, $i = 1, \dots, s(k)$, *and* $\operatorname{supp} q_0(f) \cap \mathfrak{N}(k) = \emptyset$. *Moreover, if $f \in (X)^d$, then each $q_i(f) \in (X)^{d-d_i}$ (where $(X)^\ell$ means $\underline{k}[[X]]$ if $\ell \le 0$).*

We recall that if J is an ideal in $\underline{k}[[X]]$, then the Hilbert-Samuel function of $\underline{k}[[X]]/J$ is given by $H_{\underline{k}[[X]]/J}(\ell) = \dim_{\underline{k}} \underline{k}[[X]]/(J + (X)^{\ell+1})$.

Corollary 7.4. *Let $k \in \mathbb{N}$; assume (1) and (2_k). Then $H_{\underline{k}[[X]]/(f_1,\dots,f_{s(k)})} \le H_{\mathfrak{N}(k)}$.*

Lemma 7.5. *Assume (1), (2). Then property (3) is equivalent to each of the following:*

(a) $I \oplus \underline{k}[[X]]^{\mathfrak{N}} = \underline{k}[[X]]$, where $\underline{k}[[X]]^{\mathfrak{N}} := \{f \in \underline{k}[[X]] : \operatorname{supp} f \cap \mathfrak{N} = \emptyset\}$.

(b) $H_{\underline{k}[[X]]/I} = H_{\mathfrak{N}}$.

Proof. By 7.3, (3) $\Leftrightarrow$ (a), and (b) is equivalent to the condition $I \oplus \underline{k}[[X]]^{\mathfrak{N}} = \underline{k}[[X]] \bmod (X)^{\ell+1}$, for all ℓ. Therefore, (a) $\Rightarrow$ (b). To see (b) $\Rightarrow$ (3), let $f \in I$ and write $f = \sum_{i=1}^{s} q_i(f) f_i + q_0(f)$ according to 7.3; then $q_0(f) \in (X)^{\ell+1}$, for all ℓ, so $q_0(f) = 0$. $\Box$

Remark 7.6. Assume properties (1)–(3) of (7.2). Then the initial forms in f_i satisfy (1), (2) automatically. Since $H_{\underline{k}[[X]]/I} = H_{\underline{k}[[X]]/\operatorname{in} I}$, it follows from (3) and Lemma 7.5 that the $\operatorname{in} f_i$ satisfy property (3) with respect to $\operatorname{in} I$; in particular, the $\operatorname{in} f_i$ generate in I. (in I denotes the ideal generated by $\operatorname{in} f$, for all $f \in I$.) Moreover, if properties (4), (5) are satisfied, then the $\operatorname{in} f_i$ satisfy these properties as well.

For each $d \in \mathbb{N}$, let j_0^d denote the canonical projection $\underline{k}[[X]] \to \underline{k}[[X]]/(X)^{d+1}$.

Lemma 7.7. *Assume (1). Let $k \in \mathbb{N}$. Then property (2_k) is equivalent to each of the following conditions:*

$$(a) \qquad \begin{array}{ll} X^\beta (\operatorname{in} f_i)(X) , & \beta \in \Box_i,\ |\beta| \le d - d_i,\ i = 1, \dots, s(k) , \\ X^\gamma , & \gamma \notin \mathfrak{N}(k),\ |\gamma| \le d , \end{array}$$

form a basis of the vector space $\underline{k}[[X]]/(X)^{d+1}$, for each $d \in \mathbb{N}$.

$$(b) \qquad \begin{array}{ll} X^\beta j_0^{d-|\beta|} f_i , & \beta \in \Box_i,\ |\beta| \le d - d_i,\ i = 1, \dots, s(k) , \\ X^\gamma , & \gamma \notin \mathfrak{N}(k),\ |\gamma| \le d , \end{array}$$

form a basis of $\underline{k}[[X]]/(X)^{d+1}$, for each $d \in \mathbb{N}$.

Moreover, suppose that $\mathfrak{N}(k) = \mathbb{N}^{n-q} \times \mathfrak{N}^\bullet$, where $\mathfrak{N}^\bullet \subset \mathbb{N}^q$. (Thus, for each $i = 1, \dots, s(k)$, $\Box_i = \mathbb{N}^{n-q} \times \Box_i^\bullet$, where $\Box_i^\bullet \subset \mathbb{N}^q$.) Write $X = (U, V) = (U_1, \dots, U_{n-q}, V_1, \dots, V_q)$. Then each of the conditions above is equivalent to:

$$(c)\qquad \begin{array}{ll} V^\beta(\mathrm{in}\, f_i)(0,V)\,, & \beta\in\Box_i^\bullet,\ |\beta|\le d-d_i,\ i=1,\dots,s(k),\\ V^\gamma\,, & \gamma\notin\mathfrak{N}^\bullet,\ |\gamma|\le d,\end{array}$$

form a basis of $\underline{k}[[V]]/(V)^{d+1}$*, for each* $d\in\mathbb{N}$.

Proof. Obviously, $(2_k)\Leftrightarrow$ (a). Consider the square matrices with entries in $\underline{k}$ whose columns are the elements listed in (a) or (b) written in components with respect to the standard monomial basis of $\underline{k}[[X]]/(X)^{d+1}$. These matrices differ by a factor which is a triangular matrix with 1's on the diagonal. Therefore, (a) $\Leftrightarrow$ (b). In each condition of the lemma, "form a basis of" is equivalent to "span", by dimension considerations.

(a) $\Rightarrow$ (c): Let $f(V)$ be a polynomial in V of degree $\le d$. By (a),

$$f(V)\ =\ \sum_{i=1}^{s(k)} q_i(U,V)(\mathrm{in}\, f_i)(U,V)+q_0(U,V)\ ,$$

where each q_i is a polynomial of degree $\le d-d_i$, $\mathrm{supp}\, q_i\subset\Box_i$, $i=1,\dots,s(k)$, and $\mathrm{supp}\, q_0\cap\mathfrak{N}(k)=\emptyset$. Set $U=0$ to obtain (c).

(c) $\Rightarrow$ (a): Let $f(U,V)$ be a polynomial of degree $\le d$. We argue by induction on the degree e of f with respect to V. Write $f(U,V)=\sum_{\substack{\alpha\in\mathbb{N}^q\\|\alpha|\le e}} c_\alpha(U)V^\alpha$. Expressing each V^α in terms of the basis of $\underline{k}[[V]]/(V)^{|\alpha|+1}$ given by (c), we get

$$f(U,V)\ =\ \sum_{i=1}^{s(k)} c_i(U,V)(\mathrm{in}\, f_i)(0,V)+c(U,V)\ ,$$

where each $c_i(U,V)=c_i(X)$ is a linear combination of monomials X^β, $|\beta|\le d-d_i$, $\beta\in\Box_i$, and $c(U,V)$ is a linear combination of X^γ, $|\gamma|\le d$, $\gamma\notin\mathfrak{N}(k)$. Write

$$\begin{aligned} f(U,V)=&\sum_{i=1}^{s(k)} c_i(U,V)(\mathrm{in}\, f_i)(U,V)+c(U,V)\\ &+\sum_{i=1}^{s(k)} c_i(U,V)\big((\mathrm{in}\, f_i)(0,V)-(\mathrm{in}\, f_i)(U,V)\big);\end{aligned}$$

the result follows by induction since the last sum has degree $<e$ in V. $\Box$

Remark 7.8. Assume (1) and (2). It follows by dimension considerations from Lemma 7.7 that none of the $(\mathrm{in}\, f_i)(0,V)$ vanish. (Therefore each has order d_i.)

The Hilbert-Samuel function and the equimultiple locus. Let I denote an ideal in $\underline{k}[[X]]=\underline{k}[[X_1,\dots,X_n]]$. In this subsection, we will prove that if $f_i(X)\in I$, $i=1,\dots,s$, satisfy properties (1)–(5) of (7.2), then (formally) the equimultiple locus of the f_i coincides with the Hilbert-Samuel stratum of $\underline{k}[[X]]/I$.

Suppose that $f_i(X)\in\underline{k}[[X]]$, $i=1,\dots,s$. Set $d_i=\mu(f_i)$, for each i.

Definition 7.9. *For each* $k \in \mathbb{N}$, *let* $I^k_{S_{(f)}} \subset \underline{k}[[X]]$ *denote the ideal generated by all formal derivatives* $D^\alpha f_i$, $|\alpha| < \min(d_i, k+1)$, $i = 1, \ldots, s$.

Obviously, $I^k_{S_{(f)}} \subset I^{k+1}_{S_{(f)}}$ for all k, with equality when $k+1 \geq \max d_i$. In the latter case, we will say that $I^k_{S_{(f)}}$ is the ideal of the "formal equimultiple locus" of the f_i.

Write $A = \underline{k}[[X]]$ and let $Y = (Y_1, \ldots, Y_n)$ be indeterminates. Let $k \in \mathbb{N}$. Every $f \in A$ induces an element $(j^k_X f)(Y) \in A[[Y]]/(Y)^{k+1}$; namely,

$$(j^k_X f)(Y) := f(X+Y) \mod (Y)^{k+1} = \sum_{|\alpha| \leq k} \frac{D^\alpha f(X)}{\alpha!} Y^\alpha \mod (Y)^{k+1}.$$

Let $J^k I$ denote the ideal in $A[[Y]]/(Y)^{k+1}$ generated by $(j^k_X f)(Y)$, for all $f \in I$. If I is generated by $f_i(X)$, $i = 1, \ldots, s$, then, as an A-submodule of $A[[Y]]/(Y)^{k+1}$, $J^k I$ is clearly generated by $Y^\beta (j^k_X f_i)(Y)$, $i = 1, \ldots, s$, $|\beta| \leq k$.

Definition 7.10. *For each* $k \in \mathbb{N}$, *let* $I^k_S \subset \underline{k}[[X]]$ *be the ideal* $\sum_{j \leq k} \overset{\circ}{\mathrm{I}}{}^j_S$, *where* $\overset{\circ}{\mathrm{I}}{}^k_S$ *denotes the local flattener of the* A-*module* $F = (A[[Y]]/(Y)^{k+1})/J^k I$.

The chain of inclusions $I^k_S \subset I^{k+1}_S$ stabilizes, of course, since A is Noetherian.

The "local flattener" of F means the smallest ideal H in A such that $F \otimes_A A/H$ is a flat (A/H)-module. We will never need this idea, so we avoid using it (or showing that H exists!) by giving an alternative explicit definition of $\overset{\circ}{\mathrm{I}}{}^k_S$ (Definition 7.13 below).

Remark 7.11. Suppose that R is a ring (commutative with 1) and that J is a submodule of a free module R^q. Let $E = R^q/J$, and let $R^q \to E$ be the canonical projection. Consider any exact sequence $R^p \xrightarrow{B} R^q \longrightarrow E \longrightarrow 0$; i.e., the columns $b_1, \ldots, b_p$ of B (regarded as a $q \times p$ matrix with entries in R) form a set of generators of J. Let $r \in \mathbb{N}$. Then the ideal H in R generated by the minors of B of order r is independent of the choice of p and B: Suppose that B' is a $q \times p'$ matrix whose columns $b'_1, \ldots, b'_{p'}$ generate J. Then each $b_i = \sum_j a_{ij} b'_j$, where the $a_{ij} \in R$. Hence $H \subset H'$. Likewise, $H' \subset H$.

Let $k \in \mathbb{N}$. We identify $A[[Y]]/(Y)^{k+1}$ with A^q, where $q = \#\{\alpha \in \mathbb{N}^n : |\alpha| \leq k\}$, by means of the standard monomial basis $\{Y^\alpha : |\alpha| \leq k\}$. Suppose that $f_i(X)$, $i = 1, \ldots, s$, form a set of generators of I; $d_i = \mu(f_i)$ for each i. Consider the presentation

$$A^p \xrightarrow{B} A^q \longrightarrow F \longrightarrow 0 \tag{7.12}$$

of the A-module $F = (A[[Y]]/(Y)^{k+1})/J^k I$, where B is the matrix with entries in A whose columns (as elements of $A[[Y]]/(Y)^{k+1}$) are

$$Y^\beta (j^k_X f_i)(Y) = \sum_{|\alpha| \leq k} \frac{D^{\alpha-\beta} f_i(X)}{(\alpha-\beta)!} Y^\alpha \mod (Y)^{k+1}, \qquad i = 1, \ldots, s, \; |\beta| \leq k.$$

($D^{\alpha-\beta} = 0$ unless $\alpha \geq \beta$ in the usual partial ordering of $\mathbb{N}^n$). Thus the columns of B are indexed by (i,β), $i = 1,\dots,s$, $|\beta| \leq k$, and the rows are indexed by α, $|\alpha| \leq k$.

By evaluation $X = 0$, B induces $\underline{k}^p \xrightarrow{B(0)} \underline{k}^q$. (In $B(0)$, the column $Y^\beta(j_0^k f_i)(Y)$ is zero unless $|\beta| \leq k - d_i$.) Set $r_k = \operatorname{rank} B(0)$. Clearly, $H_{A/I}(k) = q - r_k$.

Definition 7.13. *For each $k \in \mathbb{N}$, we can define $\overset{\circ}{\mathrm{I}}{}^k_S$ as the ideal in A generated by the minors of B of order $r_k + 1$.*

By Remark 7.11, this definition is independent of the presentation (7.12); in particular, independent of the choice of generators of I. The ideal I_S^k defines the "formal Hilbert-Samuel stratum" of I when k is large enough.

Theorem 7.14. *(cf. [BM4, Theorem 5.3.1]). Let I be an ideal in $\underline{k}[[X]] = \underline{k}[[X_1,\dots,X_n]]$, and let $\mathfrak{N} \in \mathscr{D}(n)$. Let $f_i(X) = f_i(W,Z) \in I$, $i = 1,\dots,s$, be elements satisfying properties (1)–(5) of (7.2), where $K \geq \max d_i - 1$ ($d_i = \mu(f_i)$). If $k \geq \max d_i - 1$, then $I^k_{S(f)} = I^k_S$. (The inclusion $I^k_{S(f)} \subset I^k_S$ does not require property (3).)*

Proof. Let B denote the matrix above: The columns of B are the $Y^\beta j_X^k f_j(Y)$, $j = 1,\dots,s$, $|\beta| \leq k$. (More precisely, the components of the column vectors are the coefficients of the monomials Y^α in these elements.) B has column index (j,β) and row index $\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$. Consider the minor of B of order $r_k = \operatorname{rank} B(0)$ determined by the columns $Y^\beta j_X^k f_j(Y)$, where $\beta \in \Box_j$, $|\beta| \leq k - d_j$, and the rows indexed by $\alpha \in \mathfrak{N}$, $|\alpha| \leq k$ (cf. Lemma 7.7). (Only columns such that $d_j \leq k$ are involved.)

We claim that this minor is a unit (i.e., nonzero when $X = 0$). To see this, consider the following block matrix with entries in $\underline{k}[[X]]$. (The columns are indicated along the top, and the rows are labelled at the left.)

	$Y^\beta j_X^k f_j(Y)$, $\beta \in \Box_j, \lvert\beta\rvert \leq k - d_j$	Y^γ, $\gamma \notin \mathfrak{N}, \lvert\gamma\rvert \leq k$
$\alpha \in \mathfrak{N}$, $\lvert\alpha\rvert \leq k$	C	0
$\alpha \notin \mathfrak{N}$, $\lvert\alpha\rvert \leq k$		identity

The minor we are interested in is the determinant of the upper left block C. The entire matrix here is invertible by properties (1), (2) of (7.2). The upper right block is zero, and the lower right is the identity (provided that the corresponding rows and columns are ordered lexicographically with respect to α or γ). This establishes the claim.

Lemma 7.15. *Suppose that $k < d_i$, where $1 \le i \le s$. Then, for all $\gamma \notin \mathfrak{N}$, $|\gamma| \le k$,*

$$D^\gamma f_i \;\in\; \overset{\circ}{\mathrm{I}}{}^k_S + (D^\alpha f_i :\; \alpha \in \mathfrak{N},\; |\alpha| \le k)\,.$$

Proof. Consider the minor φ of B of order $r_k + 1$ determined by adjoining to the submatrix C above, the column $j^k_X f_i(Y)$ and the row indexed by γ (where $\gamma \notin \mathfrak{N}$, $|\gamma| \le k$). Then $\varphi \in \overset{\circ}{\mathrm{I}}{}^k_S$. But, expanding φ by cofactors along the column determined by $j^k_X f_i(Y)$, we see that $\varphi = D^\gamma f_i \times$ unit modulo the ideal $(D^\alpha f_i :\; \alpha \in \mathfrak{N},\; |\alpha| \le k)$. □

Remark 7.16. For each $\ell = 1, \ldots, p$, the ideal $(Z^\ell) \subset I^{k_\ell - 1}_{S_{(f)}}$, by property (4) of (7.2) (and the implicit function theorem).

Lemma 7.17. *For all $\ell = 1, \ldots, p$, $(Z^\ell) \subset I^{k_\ell - 1}_S$.*

Proof. By induction on ℓ. Assume that $(Z^\ell) \subset I^{k_\ell - 1}_S$ (vacuous assumption if $\ell = 0$). It suffices to show that $D^{\beta_j} f_{i(j)} \in I^{k_{\ell+1}-1}$ if $|\alpha^{i(j)}| = k_{\ell+1}$. Let $k = k_{\ell+1} - 1$. By Lemma 7.15, $D^{\beta_j} f_{i(j)} \in \overset{\circ}{\mathrm{I}}{}^k_S + (D^\alpha f_{i(j)} :\; \alpha \in \mathfrak{N},\; |\alpha| \le k)$. But, by property (5), the ideal $(D^\alpha f_{i(j)} :\; \alpha \in \mathfrak{N},\; |\alpha| \le k) \subset (Z^\ell) \subset I^{k_\ell - 1}_S \subset I^{k_{\ell+1}-1}_S$. □

To complete the proof of Theorem 7.14: Clearly, $I^0_{S_{(f)}} = I = \overset{\circ}{\mathrm{I}}{}^0_S = I^0_S$.

We first prove that, for all k, $I^k_{S_{(f)}} \subset I^k_S + (Z)$. (In particular, if $k \ge k_p - 1$, then by 7.17, $(Z) \subset I^{k_p - 1}_S \subset I^k_S$ and $I^k_{S_{(f)}} \subset I^k_S$.) By induction, we can assume that $I^{k-1}_{S_{(f)}} \subset I^{k-1}_S + (Z) \subset I^k_S + (Z)$. Hence it is enough to show that $D^\alpha f_i \in I^k_S + (Z)$ if $|\alpha| = k < d_i$. Suppose $d_i = k_{\ell+1}$. We will show that $D^\alpha f_i \in \overset{\circ}{\mathrm{I}}{}^k_S + (Z^\ell)$: If $\alpha \in \mathfrak{N}$, then $\alpha \in \mathbb{N}^{n-r} \times \mathfrak{N}(k_\ell)$ and $|\alpha| \le K$, so that $D^\alpha f_i \in (Z^\ell)$, by property (5) of (7.2). On the other hand, if $\alpha \notin \mathfrak{N}$, then $D^\alpha f_i \in \overset{\circ}{\mathrm{I}}{}^k_S + (Z^\ell)$, by 7.15 and the previous case.

Finally, we show that for all k, $I^k_S \subset I^k_{S_{(f)}}$. We first remark that $J^k I$ is generated by the elements $Y^\beta j^k_X f_i(Y)$, $\beta \in \square_i$, $|\beta| \le k$, $i = 1, \ldots, s$: By property (3), for any $f \in I$, $f(X) = \sum_{i=1}^s q_i(X) f_i(X)$, where $\operatorname{supp} q_i \subset \square_i$ for each i. Therefore, $f(X + Y) \bmod (Y)^{k+1}$ is a linear combination over $A = \underline{k}[[X]]$ of the alleged generators.

Consider the presentation of F determined by the above set of generators of $J^k I$: $A^{p'} \xrightarrow{B'} A^q \longrightarrow F \longrightarrow 0$. Here the columns of B' are indexed by (i, β), where $\beta \in \square_i$, $|\beta| \le k$, $i = 1, \ldots, s$, and the rows by $\alpha \in \mathbb{N}^n$, $|\alpha| \le k$. By induction, it is enough to show that $\overset{\circ}{\mathrm{I}}{}^k_S \subset I^k_{S_{(f)}}$. Since $H_{A/I} = H_{\mathfrak{N}}$, $r_k = q - H_{A/I}(k) = \#\{\alpha \in \mathfrak{N} :\; |\alpha| \le k\}$. Therefore, there are precisely r_k columns of B' with index (i, β) such that $|\beta| \le k - d_i$. It is clearly enough to show that, for each column where $|\beta| > k - d_i$, all entries belong to $I^k_{S_{(f)}}$. But these entries are the coefficients of $Y^\beta f_i(X+Y) \bmod (Y)^{k+1}$; in other words, the coefficients of $f_i(X+Y) \bmod (Y)^{k-|\beta|+1}$, or the formal derivatives of $f_i(X)$ of orders $\leq k-|\beta| < d_i$; the latter belong to $I^k_{S_{(f)}}$, by definition. □

Remark 7.18. If we use the standard basis of I for the generators $f_i(X)$, then the proof of Theorem 7.14 (becomes a little simpler and) shows that $I^k_{S_{(f)}} = I^k_S$ for all k (cf. [BM4, Sect. 5.3]). The weaker statement as formulated is the price of using generators which provide a (regular) presentation of the Hilbert-Samuel function.

Presentation of the Hilbert-Samuel function. Let I denote an ideal in $\underline{k}[[X]] = \underline{k}[[X_1,\ldots,X_n]]$, let $\mathfrak{N} \in \mathscr{D}(n)$, and let $f_i(X)$, $i = 1,\ldots,s$, denote a set of generators of I satisfying properties (1)–(5) of (7.2), where $K \geq \max|\alpha^i|-1$. (We use the notation of (7.1).) We will show that the $f_i(X)$ determine a codimension 0 presentation of the Hilbert-Samuel function $H_{\underline{k}[[X]]/I}$ in the sense of Definition 6.2, *formally*.

If $k \geq \max|\alpha^i| - 1$, then $I^k_S = I^{k+1}_S$, by 7.14 (and 7.9); write $I^k_S = I_S$. By 7.17, $(Z) \subset I_S$; I_S is the "ideal of the formal Hilbert-Samuel stratum S" of I.

The "strict transform of I by a blowing-up σ with smooth centre C" makes sense formally: Let I_C denote an ideal $(X_\ell : \ell \in J)$, for some $J \subset \{1,\ldots,n\}$. Say $t = \#J$. Then the formal blowing-up along I_C has fibre (over $X = 0$) given by the $(t-1)$-dimensional projective space $\mathbb{P}^{t-1} =$ "$\sigma^{-1}(0)$" of lines through 0 in $\{x \in \underline{k}^n : x_\ell = 0 \text{ if } \ell \notin J\}$. Let $\xi = [\xi_\ell : \ell \in J] \in \mathbb{P}^{t-1}$ (in homogeneous coordinates). If $\xi_k \neq 0$, say $\xi_k = 1$, then σ can be defined at ξ by the homomorphism σ^*_ξ: $\underline{k}[[X]] \to \underline{k}[[X']]$, where $X' = (X'_1,\ldots,X'_n)$, given by the formal substitution

$$X_\ell = X'_\ell\ , \text{ if } \ell \notin J \text{ or } \ell = k, \quad X_\ell = X'_k(\xi_\ell + X'_\ell), \text{ if } \ell \in J\backslash\{k\}\ .$$

Let $f \in \underline{k}[[X]]$. We write $\sigma^*_\xi(f) = f \circ \sigma$. We define the *order* $\mu_{I_C}(f)$ *of* f *along* I_C as $\max\{d : f \in I^d_C\}$. The *strict transform of* I *by* σ is defined, for each $\xi \in \mathbb{P}^{t-1}$, by the ideal $I'_\xi \subset \underline{k}[[X']]$ generated by $f' = (X'_{\mathrm{exc}})^{-d} f \circ \sigma$, where $d = \mu_{I_C}(f)$, for all $f \in I$. ($X'_{\mathrm{exc}} := X'_k$ in the substitution formula above.) We write $\mu_\xi(f')$ for the order of f'.

Theorems 7.20 and 7.21 describe the transforms of I by admissible and exceptional blowings-up (cf. (4.3), (4.4)). The effect of a morphism of type (ii) is trivial, so it will follow that the $f_i(X)$ determine a codimension 0 presentation of the Hilbert-Samuel function. Write $H_I = H_{\underline{k}[[X]]/I}$ for brevity. For each $i = 1,\ldots,s$, write

$$f_i(X) = f_i(W,Z) = \sum_{\gamma\in\mathbb{N}^r} c_{i\gamma}(W)Z^\gamma\ . \tag{7.19}$$

First suppose $I_C \supset I_S$ (i.e., σ is an admissible blowing-up). Then $(Z) \subset I_C$. By 7.14, $d_i := \mu(f_i) = \mu_{I_C}(f_i)$, $i = 1,\ldots,s$. So by Lemma 5.1, $\mu_\xi(f'_i) \leq d_i$ for each i.

Theorem 7.20. (cf. [H3, Sect. 6, Prop. 1], [BM4, Theorem 7.3]). *Let* $\xi \in \mathbb{P}^{t-1} = \sigma^{-1}(0)$. *Then:*

(1) $H_{I'_\xi} \leq H_I$ *(cf. [Ben]).*

(2) $H_{I'_\xi} = H_I$ *if and only if* $\mu_\xi(f_i') = d_i$, $i = 1, \ldots, s$.

(3) Let $\xi = [\xi_\ell : \ell \in J]$*, as above. In case (2), there exists* $k \leq n - r$ *such that* $k \in J$ *and* $\xi_k \neq 0$*; say* $\xi_k = 1$*. Let* I'_{S_ξ} *denote the ideal of the formal Hilbert-Samuel stratum of* I'_ξ*. Then (using the notation of (7.19) and the substitution formula above):*

(i) $(Z') \subset I'_{S_\xi}$*, where* $X' = (W', Z') = (W'_1, \ldots, W'_{n-r}, Z'_1, \ldots, Z'_r)$.

(ii) Each $f_i'(W', Z') = \sum_{\gamma \in \mathbb{N}^r} c'_{i\gamma}(W')(Z')^\gamma$*, where* $c'_{i\gamma} = (W'_{\text{exc}})^{-(d_i - |\gamma|)} c_{i\gamma} \circ \sigma$.

(iii) The f_i' *satisfy properties (1)–(5) of (7.2) with respect to the ideal* $I' = I'_\xi$ *and the diagram* $\mathfrak{N}' = \mathfrak{N}$.

Secondly, assume that $r \leq n - 2$ and that $I_C = (W_1, W_2)$. Let σ denote the formal blowing-up along I_C. (An exceptional blowing-up (4.3)(iii) has this form.)

Theorem 7.21. *Let* $\xi = [1, \xi_2] \in \mathbb{P}^1$*. Then:*

(1) I'_ξ *is generated by* $\sigma_\xi^*(I)$*, and* $(Z') \subset I'_{S_\xi}$*, where* I'_{S_ξ} *is the ideal of the formal Hilbert-Samuel stratum of* I'_ξ*.*

(2) Each $f_i'(W', Z') = (f_i \circ \sigma)(W', Z') = \sum_{\gamma \in \mathbb{N}^r} c'_{i\gamma}(W')(Z')^\gamma$*, where* $c'_{i\gamma} = c_{i\gamma} \circ \sigma$.

(3) The f_i' *satisfy properties (1)–(5) of (7.2) with respect to the ideal* $I' = I'_\xi$ *and the diagram* $\mathfrak{N}' = \mathfrak{N}$*. (In particular,* $H_{I'} = H_I$ *and* $\mu_\xi(f_i') = d_i$, $i = 1, \ldots, s$*.)*

The following assertions will be used in our proofs of Theorems 7.20 and 7.21.

Lemma 7.22. *Let* P *denote the ideal* $(X_\ell : \ell \in J)$*, for some* $J \subset \{1, \ldots, n\}$*. Suppose that* $f_i(X) \in I$, $i = 1, \ldots, s$*, satisfy properties (1)–(3) of (7.2) and that* $\mu_P(f_i) = d_i := \mu(f_i)$, $i = 1, \ldots, s$*. If* $f \in I$ *and* $d = \mu_P(f)$*, then* $\mu_P\big(q_i(f)\big) \geq d - d_i$, $i = 1, \ldots, s$ *(where the* $q_i(f)$ *are the quotients in the division formula of property (3)).*

Proof. For each i, since $\mu_P(f_i) = d_i$, $\text{in} f_i$ depends on the variables X_ℓ, $\ell \in J$, alone. Let $f \in I$ $(f \neq 0)$ and let $d = \mu_P(f)$. Set $e_i = \mu_P\big(q_i(f)\big)$, $i = 1, \ldots, s$, and $e = \min_i(e_i + d_i) < \infty$. Let $\text{in}_P g$, $g \in \underline{k}[[X]]$, denote the initial form of g as a formal expansion in X_ℓ, $\ell \in J$, with coefficients in $\underline{k}[[X_\ell : \ell \notin J]]$. We claim that

$$\sum_{\{i : e_i + d_i = e\}} \text{in}_P q_i(f) \cdot \text{in}_P f_i \neq 0 , \tag{7.23}$$

Of course, for each i, $\operatorname{supp} \text{in}_P q_i(f) \subset \Box_i$ (supp of $\text{in}_P q_i(f)$ as a power series in X) and $\text{in}_P f_i = \text{in} f_i +$ terms of higher order in X. Suppose that (7.23) is not true. Write each term in the left-hand side as a sum of its homogeneous parts with respect to X; thus $\sum (q_{i,e_i'} + \cdots)(\text{in} f_i + h_{i,d_i+1} + \cdots) = 0$, where the second subscript in each case indicates degree of homogeneity and each $q_{i,e_i'} \neq 0$. If

$e' = \min(e'_i + d_i)$, then $\sum_{\{i:e'_i+d_i=e'\}} q_{i,e'_i} \cdot \operatorname{in} f_i = 0$. Therefore all $q_{i,e'_i} = 0$ in this sum, by property (2); a contradiction. It follows from (7.23) that $d = e$ and therefore $e_i \geq d - d_i$, for all i. □

Corollary 7.24. *(cf. [BM4, Lemma 7.1]). Suppose that $f_i(X) \in I$, $i = 1, \ldots, s$, satisfy properties (1)–(3) of (7.2). Consider the strict transform of I by the formal blowing-up along $I_C = (X_\ell : \ell \in J)$, where $J \subset \{1, \ldots, n\}$, as above. If $\mu_{I_C}(f_i) = d_i := \mu(f_i)$, $i = 1, \ldots, s$, then, for all $\xi \in \sigma^{-1}(0) = \mathbb{P}^{t-1}$, the ideal I'_ξ is generated by $f'_i = (X'_{\mathrm{exc}})^{-d_i} f_i \circ \sigma$, $i = 1, \ldots, s$.*

Lemma 7.25. *(cf. [BM4, Lemma 7.5]). Let $h_i(U, V) \in \underline{k}[[U, V]]$, $i = 1, \ldots, s$, where $U = (U_1, \ldots, U_p)$, $V = (V_1, \ldots, V_q)$. Let $J_1 \subset \underline{k}[[U, V]]$ denote the ideal generated by the $h_i(U, V)$, and $J_0 \subset \underline{k}[[U, V]]$ the ideal generated by the $h_i(0, V)$. Then $H_{J_1} \leq H_{J_0}$.*

Proof. For each $\lambda \in \underline{k}$, let $J(\lambda) \subset \underline{k}[[U, V]]$ denote the ideal generated by the $h_i(\lambda U, V)$. If $\lambda \neq 0$, then $(U, V) \mapsto (\lambda U, V)$ induces an automorphism of $\underline{k}[[U, V]]$ taking $J(1)$ onto $J(\lambda)$; in particular, $H_{J(\lambda)} = H_{J(1)}$. Therefore, $H_{J(1)} \leq H_{J(0)}$, by semicontinuity of the Hilbert-Samuel function (cf. Remarks 9.1). But $J(1) = J_1$ and $J(0) = J_0$. □

Lemma 7.26. *Let $h_i(U, V) \in \underline{k}[[U, V]]$, $i = 1, \ldots, s$, where $U = (U_1, U_2)$, $V = (V_1, \ldots, V_q)$. Let J denote the ideal generated by the $h_i(U, V)$, and let J' denote the ideal generated by the $h_i\big(U_1, U_1(c + U_2), V\big)$, where $c \in \underline{k}$. Then $H_{J'} \geq H_J$.*

Proof. If $\lambda \in \underline{k}$, let $J(\lambda)$ be the ideal generated by the $h_i\big(U_1, \lambda U_2 + (1 - \lambda)U_1(c + U_2), V\big)$. Then $J(0) = J'$, $J(1) = J$. If $\lambda \neq 0$, then the substitution $(U_1, U_2, V) \mapsto \big(U_1, \lambda U_2 + (1 - \lambda)U_1(c + U_2), V\big)$ induces an automorphism of $\underline{k}[[U, V]]$ taking $J = J(1)$ to $J(\lambda)$. Therefore, $H_{J(\lambda)} = H_J$, for all $\lambda \neq 0$. By semicontinuity, $H_{J'} = H_{J(0)} \geq H_J$. □

Proof of Theorem 7.20. We can assume that $W = (T, Y)$, $Y = (Y_1, \ldots, Y_q)$, $T = (T_1, \ldots, T_{n-q-r})$ and that I_C is the ideal (Y, Z). We write

$$f_i(T, Y, Z) = \sum_{\gamma \in \mathbb{N}^r} c_{i\gamma}(T, Y) Z^\gamma , \qquad i = 1, \ldots, s .$$

For each i and γ, let $c_{i\gamma, d_i - |\gamma|}$ denote the homogeneous part of $c_{i\gamma}(T, Y)$ of order $d_i - |\gamma|$. For each i, since $\mu_{I_C}(f_i) = d_i = \mu(f_i)$, if $|\gamma| < d_i$ then $c_{i\gamma, d_i - |\gamma|} = c_{i\gamma, d_i - |\gamma|}(Y)$ depends only on Y, and

$$(\operatorname{in} f_i)(T, Y, Z) = \sum_{|\gamma| \leq d_i} c_{i\gamma, d_i - |\gamma|}(Y) Z^\gamma .$$

Let $J = \mathrm{in}\, I$. By Remark 7.6, J is the ideal generated by the $\mathrm{in}\, f_i$. Let $\eta \in \underline{k}^q$, $\zeta \in \underline{k}^r$, and let $J_{(\eta,\zeta)} \subset \underline{k}[[X]] = \underline{k}[[T, Y, Z]]$ denote the ideal generated by the

$$(\mathrm{in}\, f_i)(T, \eta + Y, \zeta + Z) \;=\; \sum_{|\gamma| \le d_i} c_{i\gamma, d_i - |\gamma|}(\eta + Y)(\zeta + Z)^\gamma \;.$$

If $\lambda \in \underline{k}$, let I_λ be the ideal generated by the $(\mathrm{in}\, f_i)(T, \lambda\eta + Y, \lambda\zeta + Z)$. If $\lambda \neq 0$, then $(T, Y, Z) \mapsto (\lambda T, \lambda Y, \lambda Z)$ induces an automorphism of $\underline{k}[[X]]$ taking I_λ to $J_{(\eta,\zeta)}$; therefore (as in 7.25), $H_J = H_{I_0} \ge H_{I_\lambda} = H_{J_{(\eta,\zeta)}}$. Then the following are equivalent:

(7.27) (1) $(\mathrm{in}\, f_i)(T, Y, \zeta + Z)$ has order $< d_i$, for some i;
(2) $H_{J_{(0,\zeta)}} < H_J$
(3) $\zeta \neq 0$.

Indeed, the $\mathrm{in}\, f_i$ satisfy (7.2) with respect to $\mathrm{in}\, I$, by Remark 7.6, so that (1) $\Leftrightarrow$ (2) by Theorem 7.14. (2) $\Rightarrow$ (3) since $J_{(0,0)} = J$, and (3) $\Rightarrow$ (1) by property (4) of (7.2).

Consider $\xi \in \sigma^{-1}(0) = \mathbb{P}^{t-1}$ (where $t = q + r$). By Corollary 7.24, for any choice of homomorphism σ_ξ^*: $\underline{k}[[X]] \to \underline{k}[[X']]$ as above, the strict transform $I' = I'_\xi$ of I is generated by the $f_i' = (X'_{\mathrm{exc}})^{-d_i} f_i \circ \sigma$, and the strict transform $J' = J'_\xi$ of J is generated by the $(\mathrm{in}\, f_i)'$. Write $\xi = [\eta, \zeta] = [\eta_1, \ldots, \eta_q, \zeta_1, \ldots, \zeta_r]$ in homogeneous coordinates.

Case I. First suppose that $\eta_k \neq 0$ for some k; say $\eta_1 = 1$. Write $X' = (S, U, V) = (S_1, \ldots, S_{n-q-r}, U_1, \ldots, U_q, V_1, \ldots, V_r)$, so that σ_ξ^* can be defined by the formal substitution

$$T = S, \quad Y_1 = U_1, \quad Y_k = U_1(\eta_k + U_k)\,,\; k = 2, \ldots, q, \quad Z = U_1(\zeta + V)\,.$$

Write $\widetilde{\eta} = (\eta_2, \ldots, \eta_q)$ and $\widetilde{U} = (U_2, \ldots, U_q)$. For each $i = 1, \ldots, s$, we have

$$\begin{aligned} (7.28)\quad f_i'(S, U_1, \widetilde{U}, V) \;&:=\; U_1^{-d_i} f_i\big(S, U_1, U_1(\widetilde{\eta} + \widetilde{U}), U_1(\zeta + V)\big) \\ &=\; \sum_{\gamma \in \mathbb{N}^r} U_1^{-(d_i - |\gamma|)} c_{i\gamma}\big(S, U_1, U_1(\widetilde{\eta} + \widetilde{U})\big)(\zeta + V)^\gamma \;; \end{aligned}$$

$$(7.29)\quad (\mathrm{in}\, f_i)'(S, U, V) \;=\; \sum_{|\gamma| \le d_i} c_{i\gamma, d_i - |\gamma|}(1, \widetilde{\eta} + \widetilde{U})(\zeta + V)^\gamma \;=\; f_i'(0, 0, \widetilde{U}, V)\;.$$

By (7.29) and Lemma 7.25, $H_{I'} \le H_{J'}$. Consider the isomorphism θ: $\underline{k}[[X]] \to \underline{k}[[X']]$ induced by the substitution $T = S$, $Y_1 = U_1$, $Y_k = (1 + U_1)(\eta_k + U_k) - \eta_k$, $k = 2, \ldots, q$, and $Z = (1 + U_1)(\zeta + V) - \zeta$. Then θ takes $J_{(\eta,\zeta)}$ to J'. Therefore, $H_{I'} \le H_{J'} = H_{J_{(\eta,\zeta)}} \le H_J = H_I$.

Let us write $(1')$–$(5')$ to mean (1)–(5) of (7.2) for the f_i' (with respect to I', $\mathfrak{N}' = \mathfrak{N}$). We obtain the conclusion of the theorem in Case I from the assertions:

(a) If $\mu_\xi(f_i') = d_i$, $i = 1, \ldots, s$ (i.e., $(1')$ holds), then properties $(1') - (5')$ all hold; in particular, $H_{I'} = H_I$ by Lemma 7.5.

(b) If $\mu_\xi(f_i') < d_i$, for some i, then $H_{I'} < H_I$.

For each i, if $\mu_\xi(f_i') = d_i$, then by (7.28), the summands of $f_i'(0, 0, V)$ indexed by γ with $|\gamma| < d_i$ contribute zero, so that

$$f_i'(0,0,V) \;=\; \sum_{|\gamma|=d_i} c_{i\gamma,0}(\zeta+V)^\gamma \;=\; \sum_{|\gamma|=d_i} c_{i\gamma,0}V^\gamma$$

(each $c_{i\gamma,0}$ = a constant); it follows that $f_i'(0,0,V) = (\mathrm{in} f_i')\,(0,0,V) = (\mathrm{in} f_i)(0,0,V)$. Therefore, if $\mu_\xi(f_i') = d_i$ for all i (i.e., (1′) holds), then $\zeta = 0$ (by (7.27)). Moreover, for each $\ell = 1,\ldots,p$, if $\mu_\xi(f_i') = d_i$, $i = 1,\ldots,s_\ell$, then $(2'_{k_\ell})$ follows from (2_{k_ℓ}) using 7.7.

For each $j = 1,\ldots,r$, $D^{\beta_j}(f_{i(j)}\circ\sigma) = U_1^{|\beta_j|}(D^{\beta_j}f_{i(j)})\circ\sigma = U_1^{|\beta_j|+1}g_j'$, where $g_j' = U_1^{-1}g_j\circ\sigma$ (the strict transform of $g_j = D^{\beta_j}f_{i(j)}$); thus $g_j' = D^{\beta_j}\left(f_{i(j)}\circ\sigma/U_1^{|\beta_j|+1}\right) = D^{\beta_j}f_{i(j)}'$. For each j and ℓ, $\partial g_j'/\partial V_\ell = U_1^{-1}\partial(g_j\circ\sigma)/\partial V_\ell = (\partial g_j/\partial Z_\ell)\circ\sigma$. If $\mu_\xi(f_i') = d_i$, $i = 1,\ldots,s_\ell$, then $(4'_\ell)$ follows easily from (4_ℓ); likewise, $(5'_\ell)$ follows from (5_ℓ).

Assume that (1′) holds. We prove (3′): Let $g \in I'$. We consider 3 cases:

(i) First suppose $g = f'$, where $f \in I$; say $\mu_{I_C}(f) = d$, so that $f' = f\circ\sigma/U_1^d$. Write $f = \sum q_i(f)f_i$ according to property (3). Then, by 7.22, $\mu_{I_C}\left(q_i(f)\right) \geq d - d_i$ for each i; i.e., $q_i(f)\circ\sigma$ is divisible by $U_1^{d-d_i}$, and $f' = \sum U_1^{-(d-d_i)}\left(q_i(f)\circ\sigma\right)\cdot f_i'$, as required.

(ii) If $g = U_1^e f'$, where $f \in I$, then the result follows from (i).

(iii) Consider any $g \in I'$. By Lemma 7.3 and case (ii), it suffices to show that for any $k \in \mathbb{N}$, there exist $f \in I$ and $e \in \mathbb{N}$ such that $g - U_1^e f' \in (X')^k$. Write $g = \sum a_i f_i'$ according to Corollary 7.24. Let A_i denote the Taylor polynomial of degree k of a_i, $i = 1,\ldots,s$, and set $h = \sum A_i f_i'$, so that $g - h \in (X')^k$. If $d = \max d_i$, then $U_1^{k+d}h = \sum U_1^k A_i \cdot U_1^{d-d_i}\cdot U_1^{d_i}f_i'$; clearly, each $U_1^{d-d_i}\cdot U_1^k A_i$ can be written as $b_i\circ\sigma$, so that $U_1^{k+d}h = f\circ\sigma$, where $f = \sum b_i f_i \in I$. Therefore, $f\circ\sigma = U_1^{k+d+e}f'$, for some $e \in \mathbb{N}$, and $h = U_1^e f'$, as required.

We have thus proved assertion (a) above. To prove (b): First suppose that $\mu_\xi(f_i') < d_i$, for some $i = 1,\ldots,s_1$ (i.e., for some α^i among the first block of vertices of $\mathfrak{N}$, where $d_i = k_1$). For such i, $|\exp f_i'| < k_1$; since $H_{I'} \leq H_I = H_{\mathfrak{N}}$, it follows that $H_{I'} < H_{\mathfrak{N}}$.

In general, suppose that $\mu_\xi(f_i') = d_i$, $i = 1,\ldots,s_\ell$ and $\mu_\xi(f_{i_0}') < d_{i_0} = k_{\ell+1}$, where $s_\ell < i_0 \leq s_{\ell+1}$. Then $\mathrm{supp} f_{i_0}' \cap \{\alpha : \ |\alpha| < k_{\ell+1}\}$ is nonempty and is disjoint from $\mathfrak{N}(k_\ell)$ by $(5'_\ell)$. It follows from $(2'_{k_\ell})$ and dimension considerations (cf. Corollary 7.4) that $H_{I'} < H_{\mathfrak{N}}$. This completes Theorem 7.20 in Case I.

Case II. Suppose that $\eta_k = 0$, $k = 1,\ldots,q$. Then $\zeta_j \neq 0$, for some $j = 1,\ldots,r$; say $\zeta_1 = 1$, so that σ_ξ^* can be defined by the formal substitution

$$T = S\ ,\quad Y = V_1 U\ ,\quad Z_1 = V_1\ ,\quad Z_j = V_1(\zeta_j + V_j)\ ,\ j = 2,\ldots,r\ .$$

Write $\widetilde{\zeta} = (\zeta_2,\ldots,\zeta_r)$ and $\widetilde{V} = (V_2,\ldots,V_r)$. For each $i = 1,\ldots,s$, we have

$$f_i'(S,U,V_1,\widetilde{V}) \;=\; \sum_{\gamma\in\mathbb{N}^r} V_1^{-(d_i-|\gamma|)}c_{i\gamma}(S,V_1U)(\widetilde{\zeta}+\widetilde{V})^{\widetilde{\gamma}}\ ,$$

$$(\mathrm{in} f_i)'(S,U,V) \;=\; \sum_{|\gamma|\leq d_i} c_{i\gamma,d_i-|\gamma|}(U)(\widetilde{\zeta}+\widetilde{V})^{\widetilde{\gamma}} = f_i'(0,U,0,\widetilde{V}) \tag{7.30}$$

(where $\widetilde{\gamma}$ denotes $(\gamma_2, \ldots, \gamma_r)$). By (7.30) and 7.25, $H_{I'} \leq H_{J'}$. Consider the isomorphism θ: $\underline{k}[[X]] \to \underline{k}[[X']]$ induced by the substitution $T = S$, $Y = (1+V_1)U$, $Z_1 = V_1$, and $Z_j = (1+V_1)(\zeta_j + V_j) - \zeta_j$, $j = 2, \ldots, r$. Then θ takes each $(\mathrm{in}\, f_i)(T, Y, \zeta{+}Z)$ to $(1{+}V_1)^{d_i}(\mathrm{in}\, f_i)'(S, U, V)$, hence takes the ideal $J_{(0,\zeta)}$ to J'. By (7.27), $(\mathrm{in}\, f_i)'(S, U, V)$ has order $< d_i$, for some i, and $H_{J_{(0,\zeta)}} < H_J$. Therefore, $\mu_\xi(f_i') < d_i$, for some i (by (7.30)), and $H_{I'} \leq H_{J'} = H_{J_{(0,\zeta)}} < H_J = H_I$. This completes the proof of 7.20. □

Proof of Theorem 7.21. For each $i = 1, \ldots, s$, we write

$$f_i(W, Z) = \sum_{\gamma \in \mathbb{N}^r} c_{i\gamma}(W) Z^\gamma$$

as before, so that (as in the proof of Theorem 7.20)

$$(\mathrm{in}\, f_i)(W, Z) = \sum_{|\gamma| \leq d_i} c_{i\gamma, d_i - |\gamma|}(W) Z^\gamma .$$

Write $X' = (W', Z') = (W_1', \ldots, W_{n-r}', Z_1', \ldots, Z_r')$, so that σ_ξ^*: $\underline{k}[[X]] \to \underline{k}[[X']]$ can be defined by the formal substitution

$$W_1 = W_1' , \quad W_2 = W_1'(\xi_2 + W_2') , \quad W_k = W_k' , \ k = 3, \ldots, n-r , \quad Z = Z' .$$

By Remark 7.8, for each i, $(\mathrm{in}\, f_i)(0, Z)$ has order d_i; in particular, there exists γ such that $|\gamma| = d_i$ and $c_{i\gamma, d_i - |\gamma|}(W) = c_{i\gamma}(0) \neq 0$. Therefore, $\mu_{I_C}(f_i) = 0$ for all i, and I_ξ' is generated by the

$$f_i'(W', Z') = \sum_{\gamma \in \mathbb{N}^r} c_{i\gamma}\left(W_1', W_1'(\xi_2 + W_2'), W_3, \ldots\right)(Z')^\gamma .$$

Let us write $(1') - (5')$ to mean properties (1)–(5) of (7.2) for the f_i' (with respect to $I' = I_\xi'$, $\mathfrak{N}' = \mathfrak{N}$). By the preceding remarks, $\mu_\xi(f_i') = d_i$ for all i; i.e., $(1')$ holds. Since $(\mathrm{in}\, f_i')(0, Z) = (\mathrm{in}\, f_i)(0, Z)$ for each i, $(2')$ holds by Lemma 7.7, and $(4')$, $(5')$ follow trivially from (4), (5). By Lemma 7.26, $H_{I'} \geq H_I$. But $H_{I'} \leq H_{\mathfrak{N}} = H_I$, by Corollary 7.4. Therefore, $H_{I'} = H_{\mathfrak{N}}$ and property $(3')$ follows from Lemma 7.5. □

8. The stabilization theorem

We say that $\mathfrak{N} \in \mathscr{D}(n)$ is a *monotone* diagram if

$$(\alpha_1, \ldots, \alpha_n) \in \mathfrak{N} \ \Rightarrow \ (\alpha_1, \ldots, \underset{\substack{i'\text{th} \\ \text{place}}}{0}, \ldots, \underset{\substack{j'\text{th} \\ \text{place}}}{\alpha_i + \alpha_j}, \ldots, \alpha_n) \in \mathfrak{N}$$

whenever $i < j$ (cf. [H3]). Suppose $\mathfrak{N}$ is monotone. Let $\alpha^1, \ldots, \alpha^s$ be the vertices of $\mathfrak{N}$, in ascending order. Set $\Delta_i = \alpha^i + \Box_i$, $i = 1, \ldots, s$, and $\Box_0 = \mathbb{N}^n \backslash \mathfrak{N}$, as in Sect. 3.

Suppose A is a commutative ring with identity and $H_i(Y) \in A[Y] = A[Y_1, \ldots, Y_n]$ is a homogeneous polynomial of degree $d_i = |\alpha^i|$, $i = 1, \ldots, s$. For each $\ell \in \mathbb{N}$, set

$$\mathscr{P}(\ell) = \left\{ \begin{array}{lll} Y^\beta H_i(Y)\ , & \beta \in \square_i\ , & |\beta| = \ell - d_i\ , \quad i = 1, \ldots, s\ , \\ Y^\gamma\ , & \gamma \in \square_0\ , & |\gamma| = \ell \end{array} \right\} .$$

Of course, $\mathscr{P}(\ell)$ is a basis of the free module $(Y)^\ell/(Y)^{\ell+1}$ of homogeneous polynomials over A of degree ℓ if and only if $\mathscr{P}(\ell)$ spans this space.

Theorem 8.1. *There exists $k(\mathfrak{N}) \in \mathbb{N}$ such that, for any commutative ring A with identity and any set of homogeneous $H_i(Y) \in A[Y]$ of degree d_i, $i = 1, \ldots, s$, if $k \geq k(\mathfrak{N})$ and $\mathscr{P}(k)$ spans $(Y)^k/(Y)^{k+1}$, then $\mathscr{P}(\ell)$ spans $(Y)^\ell/(Y)^{\ell+1}$, for all $\ell \geq k$.*

Theorem 8.1 will be used in this article only in the case that A is a field.

For each $r = 0, \ldots, n-1$, let $\mathrm{pr}_r\colon \mathbb{N}^n \to \mathbb{N}^{n-r}$ denote the projection $\mathrm{pr}_r(\alpha_1, \ldots, \alpha_n) = (\alpha_{r+1}, \ldots, \alpha_n)$. Since $\mathfrak{N}$ is monotone, $F_r := (\mathbb{N} \times \mathrm{pr}_{r+1}\mathfrak{N})\backslash \mathrm{pr}_r \mathfrak{N} \subset \mathbb{N}^{n-r}$ is a finite set, for each $r = 0, \ldots, n-1$. ($\mathbb{N} \times \mathrm{pr}_n \mathfrak{N}$ means $\mathbb{N}$.) We have $\square_0 = \bigcup_{r=0}^{n-1}(\mathbb{N}^r \times F_r)$. Clearly, $\square_0 \backslash F_0 = \bigcup_{r=1}^{n-1}(\mathbb{N}^r \times F_r)$ is unbounded in the direction of the first coordinate α_1 (unless it is empty).

Since $\mathfrak{N}$ is monotone, $\mathfrak{N} \cap (\{0\} \times \mathbb{N}^{n-r})$ is monotone, $r = 0, \ldots, n-1$.

Lemma 8.2. *Let $1 \leq r \leq n$. For each i:*

(1) If Δ_i is unbounded in the α_r-direction, then $\alpha^i = (0, \ldots, 0, \alpha^i_r, \ldots, \alpha^i_n)$.

(2) If $r \geq 2$ and Δ_i is unbounded in the α_r-direction, then Δ_i is unbounded in the α_1-direction.

Proof. (1) Otherwise, if $\beta = (0, \ldots, 0, \alpha^i_1 + \cdots + \alpha^i_r, \alpha^i_{r+1}, \ldots, \alpha^i_n)$, then $\beta \in \mathfrak{N}$ and $\beta < \alpha^i$, so that $\beta + \mathbb{N}^n$ bounds $\Delta_i \subset \alpha^i + \mathbb{N}^n$ in the α_r-direction.

(2) By (1), $\alpha^i = (0, \ldots, 0, \alpha^i_r, \ldots, \alpha^i_n)$. It is enough to show that, for each $\beta_1 \in \mathbb{N}$, if $(\beta_1, 0, \ldots, 0, \alpha^i_r, \ldots, \alpha^i_n) \in \mathfrak{N}\backslash\Delta_i$, then $(0, \ldots, 0, \alpha^i_r + \beta_1, \alpha^i_{r+1}, \ldots, \alpha^i_n) \notin \Delta_i$. Now, if $\beta = (\beta_1, 0, \ldots, 0, \alpha^i_r, \ldots, \alpha^i_n) \in \mathfrak{N}\backslash\Delta_i$, then $\beta \in (\alpha^i + \mathbb{N}^n)\backslash\Delta_i$, so that $\beta = \alpha^j + \gamma$, for some $j < i$, $\gamma \in \mathbb{N}^n$. Thus $\alpha^j = (\alpha^j_1, 0, \ldots, 0, \alpha^j_r, \ldots, \alpha^j_n)$, where $0 < \alpha^j_1 \leq \beta_1$ and $\alpha^j_\ell \leq \alpha^i_\ell$, $\ell = r, \ldots, n$. Hence $|\alpha^j| < |\alpha^i|$, since otherwise $\alpha^i < \alpha^j$ (contrary to the ordering of the vertices). It follows that $(0, \ldots, 0, \alpha^j_1 + \alpha^j_r, \alpha^j_{r+1}, \ldots, \alpha^j_n) \in \mathfrak{N}$ is of the form $\alpha^k + \delta$, for some $k < i$. Then $(0, \ldots, 0, \alpha^i_r + \beta_1, \alpha^i_{r+1}, \ldots, \alpha^i_n) \in \Delta_k$, so $\notin \Delta_i$. $\square$

Definition 8.3. *Set $k'(\mathfrak{N}) = 1 + \max\{\alpha : \ \alpha \in F_0$ or $\alpha \in \Delta_i$, for all i such that Δ_i is bounded$\}$. (Take $\max\emptyset := 0$.) Set $k(\mathfrak{N}) = \max\limits_{0 \leq r \leq n-1} k'(\mathfrak{N} \cap (\{0\} \times \mathbb{N}^{n-r}))$.*

Remarks 8.4. Suppose that $k \geq k'(\mathfrak{N})$. Then:

(1) If $\beta \in \square_i$ (where $1 \leq i \leq s$) and $|\beta| = k - d_i$, then $\square_i$ is unbounded, so that $\beta + (1, 0, \ldots, 0) \in \square_i$ (by Lemma 8.2 (2)).

(2) If $\gamma \in \square_0$ and $|\gamma| = k$, then $\gamma \in \square_0 \backslash F_0$, so that $\gamma + (1, 0, \ldots, 0) \in \square_0$.

It follows that $Y_1\mathscr{P}(k) \subset \mathscr{P}(k+1)$.

Proof of Theorem 8.1. For each $r = 0, \dots, n-1$, let $Y^r = (Y_{r+1}, \dots, Y_n)$ and let

$$\begin{aligned}\mathscr{P}(r,\ell) &:= \{P(0,Y^r) = P(0,\dots,0,Y_{r+1},\dots,Y_n) :\ P \in \mathscr{P}(\ell)\} \\ &= \left\{\begin{array}{lll} Y^\beta H_i(0,Y^r), & \beta \in \Box_i \cap (\{0\}\times\mathbb{N}^{n-r}), & |\beta| = \ell - d_i, \quad i = 1,\dots,s, \\ Y^\gamma, & \gamma \in \Box_0 \cap (\{0\}\times\mathbb{N}^{n-r}), & |\gamma| = \ell \end{array}\right\}.\end{aligned}$$

In particular, $\mathscr{P}(0,\ell) = \mathscr{P}(\ell)$. Clearly, if $\mathscr{P}(k)$ spans the space $(Y)^k/(Y)^{k+1}$ of homogeneous polynomials of degree k in $(Y_1,\dots,Y_n)$, then $\mathscr{P}(r,k)$ spans the space $(Y^r)^k/(Y^r)^{k+1}$ of homogeneous polynomials of degree k in $Y^r = (Y_{r+1},\dots,Y_n)$, $r = 0,\dots,n-1$.

Take $k \geq k(\mathfrak{N})$ and assume that $\mathscr{P}(k)$ spans $(Y)^k/(Y)^{k+1}$. It suffices to prove that $\mathscr{P}(k+1)$ spans $(Y)^{k+1}/(Y)^{k+2}$.

Lemma 8.5. *If $P(Y^r)$ is homogeneous of degree $k+1$ and divisible by Y_{r+1}, then there exists $Q(Y) = Q(Y_1,\dots,Y_n) \in \operatorname{Span}\mathscr{P}(k+1)$ such that $Q(0,Y^r) = P(Y^r)$.*

Proof. $P(Y^r)/Y_{r+1} \in \operatorname{Span}\mathscr{P}(r,k)$, by the assumption that $\mathscr{P}(k)$ spans $(Y)^k/(Y)^{k+1}$. Therefore, $P(Y^r) \in \operatorname{Span}\mathscr{P}(r,k+1)$, by Remarks 8.4 (applied to $\mathfrak{N} \cap (\{0\}\times\mathbb{N}^{n-r})$). In other words, there exists $Q(Y) \in \operatorname{Span}\mathscr{P}(k+1)$ such that $Q(0,Y^r) = P(Y^r)$. $\Box$

To complete the theorem: Let $P(Y)$ be homogeneous of degree $k+1$. Write $P = P_1(Y_1,\dots,Y_n) + P_2(Y_2,\dots,Y_n) + \cdots + P_n(Y_n)$, where, for each r, $P_r(Y_r,\dots,Y_n) = P(0,\dots,0,Y_r,\dots,Y_n) - P(0,\dots,0,Y_{r+1},\dots,Y_n)$; thus P_r is divisible by Y_r. Therefore it suffices to prove that, for each r, if $P(Y^r) = P(Y_{r+1},\dots,Y_n)$ is a homogeneous polynomial of degree $k+1$, divisible by Y_{r+1}, then $P \in \operatorname{Span}\mathscr{P}(k+1)$.

By Lemma 8.5, this is true when $r = 0$. In general, by Lemma 8.5, there exists $Q(Y) \in \operatorname{Span}\mathscr{P}(k+1)$ such that $P(Y^r) = Q(0,Y^r)$. Thus $P(Y^r) = Q(Y) + Q(0,Y^r) - Q(Y)$, but $Q(0,Y^r) - Q(Y) = \sum_{q=1}^r Q_q(Y_q,\dots,Y_n)$, where each Q_q is divisible by Y_q, so the result follows by induction on r. $\Box$

9. Semicoherent presentation of the Hilbert-Samuel function

Assume that $\mathscr{A}$ is any of the categories of local-ringed spaces over $\underline{k}$, in (0.2) (1), (2). Let M denote a manifold in $\mathscr{A}$, and let X denote a closed subspace of M.

Remarks 9.1. Let $a \in |X|$ and let $I = \widehat{\mathscr{I}}_{X,a} \subset \underline{k}[[X]]$, $X = (X_1,\dots,X_n)$, where $\widehat{\mathscr{O}}_{M,a}$ is identified with $\underline{k}[[X]]$ via the Taylor homomorphism associated to local coordinates (Definition 3.4). Definition 7.13 can be applied using a presentation (7.12) induced by generators of $\mathscr{I}_{X,a}$; thus, for each $k \in \mathbb{N}$, we get an ideal $\overset{\circ}{\mathscr{T}}{}^k_{S,a}$ in $\mathscr{O}_{M,a}$ such that $\operatorname{supp}\mathscr{O}_{M,a}/\overset{\circ}{\mathscr{T}}{}^k_{S,a} = \{x \in |X| :\ H_{X,x}(k) = H_{X,a}(k)\}$ (as a germ at a). If we formulate 7.13 in a more general way, using arbitrary rank r, we get a sheaf of ideals $\overset{\circ}{\mathscr{I}}{}^k_S(r) \subset \mathscr{O}_M$ of finite type such that $\operatorname{supp} \mathscr{O}_M / \overset{\circ}{\mathscr{I}}{}^k_S(r) = \{x \in |X| : H_{X,x}(k) \geq q - r\}$, where $q = \#\{\alpha \in \mathbb{N}^n : |\alpha| \leq k\}$; hence $H_{X,\cdot}(k)$ is Zariski-semicontinuous for each k.

The module F of (7.12) itself admits an invariant definition as the completion of the ring of germs of sections of the bundle of k-jets on M at a, modulo the ideal generated by germs of sections induced by elements of $\mathscr{I}_{X,a}$ (as an $\mathscr{O}_{M,a}$-module).

Theorem 9.2 (cf. [Ben]). *Suppose that X is an object in $\mathscr{A}$. Then the Hilbert-Samuel function $H_{X,\cdot}$ is Zariski-semicontinuous.*

Proof. Let H denote any value of the Hilbert-Samuel function. By Remarks 9.1, for each $k \in \mathbb{N}$, $\{x \in |X| : H_{X,x}(k) \geq H(k)\}$ is Zariski-closed in $|X|$. (It is clear that X need not be globally embedded.) Since X is locally Noetherian, $\{x \in |X|, H_{X,x} \geq H\}$ is Zariski-closed in $|X|$. By Lemma 9.3, $H_{X,\cdot}$ locally takes only finitely many values. $\square$

Let U be a regular chart in M with coordinates $(x_1, \ldots, x_n)$. Using the Taylor homomorphism T_a: $\mathscr{O}_{M,a} \to \underline{k}[[X]] = \underline{k}[[X_1, \ldots, X_n]]$, we associate to each $a \in U$, the diagram $\mathfrak{N}_a = \mathfrak{N}(\widehat{\mathscr{I}_{X,a}}) \in \mathscr{D}(n)$. We totally order $\mathscr{D}(n)$ as follows: To each $\mathfrak{N} \in \mathscr{D}(n)$, associate the sequence $v(\mathfrak{N})$ obtained by listing the vertices of $\mathfrak{N}$ in ascending order and completing the list to an infinite sequence by using ∞ for all the remaining terms. If $\mathfrak{N}_1, \mathfrak{N}_2 \in \mathscr{D}(n)$, we say $\mathfrak{N}_1 < \mathfrak{N}_2$ provided $v(\mathfrak{N}_1) < v(\mathfrak{N}_2)$ with respect to the lexicographic ordering of such sequences. Then every decreasing sequence in $\mathscr{D}(n)$ is finite.

Lemma 9.3. *(1) For each $a \in U$, $\{x \in U : \mathfrak{N}_x \leq \mathfrak{N}_a\}$ is Zariski-open in U.*
(2) Locally, $\mathfrak{N}_x$ has only finitely many values.

Proof. (1) For each $\alpha \in \mathbb{N}^n$, we define $N_a(\alpha) := \dim_{\underline{k}} \underline{k}[[X]]/\big((X^\beta : \beta > \alpha) + \widehat{\mathscr{I}_{X,a}}\big)$, $a \in U$, where $(X^\beta : \beta > \alpha)$ is the ideal generated by the monomials X^β, $\beta > \alpha$. Then $N_\cdot(\alpha)$ is Zariski-semicontinuous on U, for each fixed α (by an argument parallel to that of Remarks 9.1 for semicontinuity of $H_{X,x}(k) = \dim_{\underline{k}} \underline{k}[[X]]/\big((X)^{k+1} + \widehat{\mathscr{I}_{X,x}}\big)$).

If $\mathfrak{N} \in \mathscr{D}(n)$ and $\alpha \in \mathbb{N}^n$, set $N_{\mathfrak{N}}(\alpha) := \#\{\gamma \in \mathbb{N}^n \backslash \mathfrak{N} : \gamma \leq \alpha\}$. It is easy to see that if $\mathfrak{N}_0, \mathfrak{N} \in \mathscr{D}(n)$ and α_0 is the largest vertex of $\mathfrak{N}_0$, then $\mathfrak{N} \leq \mathfrak{N}_0$ if and only if either $N_{\mathfrak{N}}(\beta) \leq N_{\mathfrak{N}_0}(\beta)$ for all $\beta \leq \alpha_0$, or there exists $\alpha < \alpha_0$ such that $N_{\mathfrak{N}}(\alpha) < N_{\mathfrak{N}_0}(\alpha)$ and $N_{\mathfrak{N}}(\beta) \leq N_{\mathfrak{N}_0}(\beta)$ for all $\beta < \alpha$. If $a \in U$, then, for all $\alpha \in \mathbb{N}^n$, $N_a(\alpha) = N_{\mathfrak{N}_a}(\alpha)$ (cf. Corollary 3.20). Therefore, (1) follows from Zariski-semicontinuity of $N_\cdot(\alpha)$ for each fixed α. ("Locally Noetherian" has not been used to prove (1).)

(2) Let $a \in U$ and let V be a neighbourhood of a such that any decreasing sequence of closed subspaces of U stabilizes on V. Suppose there are infinitely many values of $\mathfrak{N}_x$, $x \in V$; then there is an infinite sequence of values $\mathfrak{N}_1 <$

$\mathfrak{N}_2 < \cdots$ (since every decreasing sequence in $\mathscr{D}(n)$ is finite). Hence $\{x : \mathfrak{N}_x > \mathfrak{N}_j\}$, $j = 1, 2, \ldots$, is a decreasing sequence of Zariski-closed subsets of U which are distinct over V. But this sequence stabilizes on V, by local Noetherianness; a contradiction. □

Theorem 9.4. *Let $a \in X$ and let $\mathfrak{N} \in \mathscr{D}(n)$. Suppose we are given:*

(1) A germ $N = N_r(a)$ at a of a regular submanifold of M of codimension r, and regular functions $w_1, \ldots, w_{n-r} \in \mathscr{O}_{M,a}$ which restrict to a coordinate system on N.

(2) $f_i(W,Z) = \sum_{\gamma \in \mathbb{N}^r} c_{i\gamma}(W) Z^\gamma \in \underline{k}[[W,Z]]$, $i = 1, \ldots, s$, where $W = (W_1, \ldots, W_{n-r})$, $Z = (Z_1, \ldots, Z_r)$ and each $W_j, Z_k \in \underline{\widehat{m}}_{M,a}$ (the maximal ideal of $\widehat{\mathscr{O}}_{M,a}$).

(We are using the notation of (7.1), (7.2).) Assume that:

(i) Each W_j is induced by w_j and $\widehat{\mathscr{I}_{N,a}} = (Z)$ (so we identify $\widehat{\mathscr{O}}_{M,a} = \underline{k}[[W,Z]]$).

(ii) Each $f_i \in I := \widehat{\mathscr{I}_{X,a}} \subset \underline{k}[[W,Z]]$, and every coefficient $c_{i\gamma}(W) = D_Z^\gamma f_i(W,0)$ is the Taylor expansion of a regular function $c_{i\gamma}(w) \in \mathscr{O}_{N,a}$ (cf. Definition 3.4).

(iii) The f_i satisfy properties (1)–(5) of (7.2) (where $K \geq \max|\alpha^i| - 1$).

Let $\mathscr{G}_1(a) = \{(c_{i\gamma}(w),\ d_i - |\gamma|) : |\gamma| < d_i,\ i = 1, \ldots, s\}$. Then $(N_r(a), \mathscr{G}_1(a), \mathscr{E}_1(a) = \emptyset)$ is a codimension r presentation of $H_{X,\cdot}$ at a (cf. Definition 6.2).

Our main aim in this section is to construct semicoherent data satisfying the hypotheses of Theorem 9.4. (See Theorem 9.6 below.) But we first show how Theorem 9.4 follows from the results in Sect. 7: We use the notation and the hypotheses of Theorem 9.4.

Remarks 9.5. (1) Suppose $k \geq \max d_i - 1$. Then $I^k_{S(f)} := (D^\alpha f_i : |\alpha| < d_i,\ i = 1, \ldots, s)$ (Definition 7.9) is generated by $Z_1, \ldots, Z_r$ and the $D^\beta c_{i\gamma}$, $|\beta| < d_i - |\gamma|$, $|\gamma| < d_i$. Thus $\underline{k}[[W,Z]]/I^k_{S(f)}$ identifies with the completion of $\mathscr{O}_{N,a}/\mathscr{I}_{S(f)(a)}$, where $\mathscr{I}_{S(f)(a)} \subset \mathscr{O}_{N,a}$ is the ideal generated by the $\partial^{|\beta|} c_{i\gamma}/\partial w^\beta$, $|\beta| < d_i - |\gamma|$, $|\gamma| < d_i$, so that $\operatorname{supp} \mathscr{O}_{N,a}/\mathscr{I}_{S(f)(a)} = \{x \in |N| : \mu_x(c_{i\gamma}) \geq d_i - |\gamma|,\ |\gamma| < d_i,\ i = 1, \ldots, s\}$.

(2) For each $k \in \mathbb{N}$, set $\mathscr{I}^k_{S,a} = \sum_{j \leq k} \overset{\circ}{\mathscr{I}}{}^j_{S,a}$. By Theorem 7.14, if $k \geq \max d_i - 1$, then $\mathscr{I}^k_{S,a} = \mathscr{I}^{k+1}_{S,a}$; say $\mathscr{I}^k_{S,a} = \mathscr{I}_{S,a}$. $\operatorname{supp} \mathscr{O}_{M,a}/\mathscr{I}_{S,a}$ is the germ $S = S_{H_X}(a)$ of $\{x \in |X| : H_{X,x} = H_{X,a}\}$. It follows from Theorem 7.14 that $S \subset N = N_r(a)$.

Proof of Theorem 9.4. Let $\sigma\colon M' \to M$ be a local blowing-up (at a) with smooth centre C, and let X', N' be the strict transforms of X, N (respectively). Let $a' \in \sigma^{-1}(a)$. If $f \in \widehat{\mathscr{O}}_{M,a}$, then the "strict transform of f", $f' = y_{\text{exc}}^{-d} f \circ \sigma$ (at a'), where $d = \mu_{I_C}(f)$ and $I_C = \widehat{\mathscr{I}_{C,a}} \subset \widehat{\mathscr{O}}_{M,a} = \underline{k}[[W,Z]]$, is defined in $\widehat{\mathscr{O}}_{M',a'}$ (up to an invertible factor).

Consider the case $C \subset S \subset N$; i.e., $I_C \supset I_S$, where $I_S = \widehat{\mathcal{I}}_{S,a}$. We can assume $w = (t, y)$, $y = (y_1, \ldots, y_q)$, $t = (t_1, \ldots, t_{n-q-r})$, where C is $\{y = 0\} \subset N$. If $a' \in N'$, then there exist $w'_1, \ldots, w'_{n-r} \in \mathcal{O}_{M',a'}$ and $W'_1, \ldots, W'_{n-r}, Z'_1, \ldots, Z'_r \in \widehat{\mathcal{O}}_{M',a'}$ satisfying the analogues of (1) and (i) in 9.4, such that $\widehat{\sigma}^*_{a'}$: $\underline{k}[[T, Y, Z]] = \widehat{\mathcal{O}}_{M,a} \to \widehat{\mathcal{O}}_{M',a'} = \underline{k}[[T', Y', Z']]$ (where $W' = (T', Y')$) is given by a formal substitution as in Case I of the proof of 7.20), and $f'_i(W', Z') = \sum_{\gamma \in \mathbb{N}^r} c'_{i\gamma}(W')(Z')^\gamma$, $i = 1, \ldots, s$, where each $c'_{i\gamma} = Y_{\text{exc}}^{-(d_i - |\gamma|)} c_{i\gamma} \circ \sigma$; of course, σ induces a homomorphism $\mathcal{O}_{N,a} \to \mathcal{O}_{N',a'}$ and each $c'_{i\gamma} = y_{\text{exc}}^{-(d_i - |\gamma|)} c_{i\gamma} \circ \sigma$ makes sense as an element of $\mathcal{O}_{N',a'}$.

Theorem 7.20 can be translated as follows: Let $a' \in \sigma^{-1}(a)$. Then:

(1) $H_{X',a'} \le H_{X,a}$.

(2) The following are equivalent: (i) $H_{X',a'} = H_{X,a}$; (ii) $\mu_{a'}(f'_i) = d_i$, $i = 1, \ldots, s$; (iii) $a' \in N'$ and $\mu_{a'}(c'_{i\gamma}) \ge d_i - |\gamma|$, $|\gamma| < d_i$, $i = 1, \ldots, s$.

(3) If $H_{X',a'} = H_{X,a}$, then the f'_i satisfy properties (1)–(5) of (7.2) with respect to $I' = \widehat{\mathcal{I}}_{X',a'}$ and $\mathfrak{N}' = \mathfrak{N}$. (In particular, $S' \subset N'$, where $S' = S_{H_{X'}}(a')$.)

Now assume $r \le n - 2$ and consider the case that C is given by $w_1 = w_2 = 0$. Clearly, $N' = \sigma^{-1}(N)$. Let $a' \in \sigma^{-1}(a)$. Then there exist $w'_1, \ldots, w'_{n-r} \in \mathcal{O}_{M',a'}$ and $W'_1, \ldots, W'_{n-r}, Z'_1, \ldots, Z'_r \in \widehat{\mathcal{O}}_{M',a'}$ satisfying the analogues of (1) and (i) in 9.4, such that $\widehat{\sigma}^*_{a'}$: $\underline{k}[[W, Z]] \to \underline{k}[[W', Z']]$ is given by a substitution as in the proof of 7.21, and $f'_i(W', Z') = \sum_{\gamma \in \mathbb{N}^r} c'_{i\gamma}(W')(Z')^\gamma$, $i = 1, \ldots, s$, where each $c'_{i\gamma} = c_{i\gamma} \circ \sigma$. Theorem 7.21 means:

(1) $X' = \sigma^{-1}(X)$.

(2) The f'_i satisfy properties (1)–(5) of (7.2) with respect to the ideal $I' = \widehat{\mathcal{I}}_{X',a'}$ and $\mathfrak{N}' = \mathfrak{N}$. (In particular, $H_{X',a'} = H_{X,a}$.)

Since the effect on $\big(N_r(a), \mathcal{G}_1(a), \mathcal{E}_1(a) = \emptyset\big)$ of a transformation of type (ii) (4.4) is trivial, we see that Theorem 9.4 is a consequence of Theorems 7.14, 7.20 and 7.21. $\square$

Let U be a regular chart in M with coordinates $(x_1, \ldots, x_n)$. Let $a_0 \in U$. We identify $\widehat{\mathcal{O}}_{M,a_0}$ with $\underline{k}[[X_1, \ldots, X_n]]$ using the Taylor homomorphism (Definition 3.4), and define $\mathfrak{N} = \mathfrak{N}(\widehat{\mathcal{I}}_{X,a_0})$. By a coordinate change, we can assume $\mathfrak{N}$ is monotone (Sect. 8) and satisfies the conditions of (7.1). We get a semicoherent presentation of the $H_{X,\cdot}$, from the following. (We use the notation of (7.1).)

Theorem 9.6. *There is a covering of M by regular coordinate charts U, each of which satisfies the following assertions. Let $a_0 \in U$ and let $\mathfrak{N} = \mathfrak{N}(\widehat{\mathcal{I}}_{X,a_0})$ (with respect to the coordinates $x = (x_1, \ldots, x_n)$). Assume that $\mathfrak{N}$ is monotone. Then we can construct:*

(1) a Zariski-open neighbourhood V of a_0 in U;

(2) a regular submanifold N of V containing a_0, defined by r elements of $\mathcal{O}(U)_V$ whose gradients are linearly independent on N;

(3) formal power series $f_i = \sum_{\gamma \in \mathbb{N}^r} c_{i\gamma} Z^\gamma$, $i = 1, \ldots, s$, in $Z = (Z_1, \ldots, Z_r)$ whose coefficients $c_{i\gamma}$ are regular functions on N induced by elements of $\mathcal{O}(U)_V$;

such that: (i) $w = (w_1, \ldots, w_{n-r})$ restricts to a regular coordinate system on N, where $x = (w, z) = (w_1, \ldots, w_{n-r}, z_1, \ldots, z_r)$.

(ii) Let $S_{(f)} = \{a \in |N| : \mu_a(c_{i\gamma}) \geq |\alpha^i| - |\gamma|$, for all $|\gamma| < |\alpha^i|, i = 1, \ldots, s\}$. Then, for all $a \in S_{(f)}$, the hypotheses of 9.4 are satisfied by $N_r(a) :=$ germ of N at a, and $f_i(W, Z) = \sum c_{i\gamma,a}(W) Z^\gamma$, $i = 1, \ldots, s$, where each $c_{i\gamma,a}(W)$ is the Taylor expansion at a of $c_{i\gamma} = c_{i\gamma}(w)$. Moreover, $H_{X,a} \leq H_{X,a_0}$ for all $a \in V$, and $S_{(f)} = S_{H_{X,a_0}} \cap V$.

Proof. We will obtain the data required by induction with respect to the blocks of vertices of $\mathfrak{N}$ of given order. (See (7.1).) To begin, we can assume there are $f_i^0 \in \mathscr{O}(U)$, $i = 1, \ldots, s$, such that the f_i^0 generate $\mathscr{I}_X$ on U and $\exp f^0_{i,a_0}(w, z) = \alpha^i$. (If $f \in \mathscr{O}(U)$ and $a \in U$, write $f_a(w, z) \in \underline{k}[[w, z]]$ for the Taylor expansion of f at a; cf. Remark 3.7). We can assume $H_{X,\cdot}$ has only finitely many values $H_{X,a}$, $a \in U$.

For each $\ell = 1, \ldots, p$, write $w^\ell = (w_1, \ldots, w_{n-r}, z_1, \ldots, z_{r-r_\ell})$ and $y^\ell = (z_{r-r_\ell+1}, \ldots, z_{r-r_{\ell-1}})$, so that $w^{\ell-1} = (w^\ell, y^\ell)$ (where $r_0 = 0$ and $w^0 = (w, z) = x$). Let $Z_1, \ldots, Z_r$ be indeterminates. For all $\ell = 1, \ldots, p$, write $Z^\ell = (Z_{r-r_\ell+1}, \ldots, Z_r)$ and $Y^\ell = (Z_{r-r_\ell+1}, \ldots, Z_{r-r_{\ell-1}})$, so that $Z^\ell = (Y^\ell, Z^{\ell-1})$ ($Z^1 = Y^1$).

Let $K \in \mathbb{N}$, $K \geq \max d_i - 1$. Put $N_0 = U$. For each $\ell = 1, \ldots, p$, we will construct:

(9.7) (1_ℓ) a Zariski-open neighbourhood V_ℓ of a_0 in U;

(2_ℓ) a regular submanifold N_ℓ of V_ℓ (of codimension r_ℓ) defined by r_ℓ elements of $\mathscr{O}(U)_{V_\ell}$ whose gradients are linearly independent on N_ℓ, such that w^ℓ restricts to a regular coordinate system on N_ℓ;

(3_ℓ) expansions $f_i^\ell(w^\ell, Z^\ell) = \sum_{\gamma \in \mathbb{N}^{r_\ell}} c^\ell_{i\gamma}(w^\ell)(Z^\ell)^\gamma$, $i = 1, \ldots, s$, where the $c^\ell_{i\gamma}(w^\ell) = (D^\gamma_{Z_\ell} f_i)(w^\ell, 0)$ are regular functions on N^ℓ induced by elements of $\mathscr{O}(U)_{V_\ell}$;

such that the following properties (9.8) (a_ℓ)–(d_ℓ) are satisfied:

(9.8) (a_ℓ) $\exp f^\ell_{i,a_0} = \alpha^i$, $i = 1, \ldots, s$, where $f^\ell_{i,a_0}(w^\ell, Z^\ell) = \sum c^\ell_{i\gamma,a_0}(w^\ell)(Z^\ell)^\gamma \in \underline{k}[[w^\ell, Z^\ell]]$ and $c^\ell_{i\gamma,a}(w^\ell)$ is the Taylor expansion of $c^\ell_{i\gamma}$ at $a \in N_\ell$.

(b_ℓ) For all $j = r - r_\ell + 1, \ldots, r - r_{\ell-1}$, let $h^\ell_j(w^{\ell-1}) = D^{\beta_j} f^{\ell-1}_{i(j)}(w^{\ell-1}, 0)$. (Recall that $w^{\ell-1} = (w^\ell, y^\ell)$. Each $\beta_j \in \{0\} \times \mathbb{N}^{r_\ell}$; D^{β_j} is a partial derivative with respect to the regular variables y^ℓ and the formal variables $Z^{\ell-1}$.) Then, in V_ℓ, $N_\ell \subset N_{\ell-1}$ is defined by $h^\ell_j(w^\ell, y^\ell) = 0$ for all j, and $\det(\partial h^\ell/\partial y^\ell)(a) \neq 0$ for all $a \in N_\ell$, where $h^\ell = (h^\ell_{r-r_\ell+1}, \ldots, h^\ell_{r-r_{\ell-1}})$.

Remark 9.9. For each $a \in N_\ell$, the formal implicit function theorem gives $h^\ell_a(w^\ell, y^\ell) = U^\ell_a(w^\ell, y^\ell)\big(y^\ell - y^\ell_a(w^\ell)\big)$, where $y^\ell_a(w^\ell) \in \underline{k}[[w^\ell]]^{r_\ell - r_{\ell-1}}$ and $U^\ell_a(w^\ell, y^\ell)$ is an invertible matrix with entries in $\underline{k}[[w^\ell, y^\ell]]$. It follows from ($b_k$), $k = 1, \ldots, \ell$, that, for each $a \in N_\ell$, there is an identification of $\widehat{\mathscr{O}}_{M,a}$ with $\underline{k}[[w^\ell, Z^\ell]]$ induced by the identification $\widehat{\mathscr{O}}_{M,a} \cong \underline{k}[[w^{\ell-1}, Z^{\ell-1}]] =$

$\underline{k}[[w^\ell, y^\ell, Z^{\ell-1}]]$ (given by induction) and the formal coordinate change $Y^\ell = y^\ell - y_a^\ell(w^\ell)$. Via this identification, $\widehat{\mathscr{I}_{N_\ell,a}} \subset \underline{k}[[w^\ell, Z^\ell]]$ is the ideal (Z^ℓ) generated by $Z_{r-r_\ell+1}, \ldots, Z_r$.

(c_ℓ) Let $a \in N_\ell$. For $i = 1, \ldots, s$, $f_{i,a}^\ell(w^\ell, Z^\ell) \in I_a^\ell$, where $I_a^\ell = \widehat{\mathscr{I}_{X,a}} \subset \underline{k}[[w^\ell, Z^\ell]]$. Moreover, for $i = 1, \ldots, s_\ell$, $f_{i,a}^\ell(w^\ell, Z^\ell) = f_{i,a}^{\ell-1}\big(w^\ell, y_a^\ell(w^\ell) + Y^\ell, Z^{\ell-1}\big)$.

Remark 9.10. Let $g_j^\ell(w^\ell, Z^\ell) = D^{\beta_j} f_{i(j)}^\ell(w^\ell, Z^\ell)$, $j = r - r_\ell + 1, \ldots, r$. (Each D^{β_j} is a derivative with respect to Z^ℓ.) It follows from (b_k) and (c_k), $k = 1, \ldots, \ell$, that each $g_{j,a}^\ell(w^\ell, Z^\ell) \in (Z^\ell)$ and $\det(\partial g_a^\ell/\partial Z^\ell)(0,0) \neq 0$, where $g^\ell = (g_{r-r_\ell+1}^\ell, \ldots, g_r^\ell)$, $a \in N_\ell$.

(d_ℓ) Let $a \in N_\ell$. If $i > s_\ell$, $\beta \in \mathfrak{N}^\ell$ and $|\beta| \leq K$, then $D_{Z^\ell}^\beta f_{i,a}^\ell(w^\ell, Z^\ell) \in (Z^\ell)$.

To construct data as above: Suppose, by induction, that for each $k = 1, \ldots, \ell-1$, we have (9.7) (1_k)–(3_k) satisfying (9.8) (a_k)–(d_k).

Define h^ℓ as in (b_ℓ) above. By (a_k) and (b_k), $k < \ell$, and the definition of the β_j in (7.1), $\det(\partial h^\ell/\partial y^\ell)(a_0) \neq 0$. Therefore, there is a Zariski-open neighbourhood V_ℓ' of a_0 in $V_{\ell-1}$, such that $\{h^\ell(w^\ell, y^\ell) = 0\}$ defines a regular submanifold $N_\ell \subset N_{\ell-1} \cap V_\ell'$ of codimension r_ℓ in V_ℓ' (the h_j^ℓ are induced by elements of $\mathscr{O}(U)_{V_\ell'}$ as in Lemma 3.5), and $\det(\partial h^\ell/\partial y^\ell)(a) \neq 0$, for all $a \in N_\ell$. Thus we have got (b_ℓ).

For each $a \in N_\ell$, we define $y_a^\ell(w^\ell)$ as in Remark 9.9. If $f(w^{\ell-1}, Z^{\ell-1}) \in \widehat{\mathscr{O}}_{M,a} = \underline{k}[[w^{\ell-1}, Z^{\ell-1}]]$, then we write $f(w^\ell, Z^\ell)$ to denote f after the identification of $\widehat{\mathscr{O}}_{M,a}$ with $\underline{k}[[w^\ell, Z^\ell]]$ via the formal change of variables $Y^\ell = y^\ell - y_a^\ell(w^\ell)$; i.e., $f(w^\ell, Z^\ell)$ means $f\big(w^\ell, y_a^\ell(w^\ell) + Y^\ell, Z^{\ell-1}\big)$. We clearly still have $\exp f_{i,a_0}^{\ell-1}(w^\ell, Z^\ell) = \alpha^i$, $i = 1, \ldots, s$ (because of the lexicographic ordering of multiindices).

Remark 9.11. Suppose $f(w^{\ell-1}, Z^{\ell-1}) = \sum_{\gamma \in \mathbb{N}^{r_{\ell-1}}} f_\gamma(w^{\ell-1})(Z^{\ell-1})^\gamma$, where each $f_\gamma(w^{\ell-1})$ is a regular function on $N^{\ell-1}$. Let $a \in N^\ell$. Consider $f_a(w^{\ell-1}, Z^{\ell-1}) \in \underline{k}[[w^{\ell-1}, Z^{\ell-1}]]$. Let $\beta \in \mathbb{N}^{r_\ell}$; say $\beta = (\delta, \gamma)$, where $\gamma \in \mathbb{N}^{r_{\ell-1}}$. Then

$$D_{Z^\ell}^\beta f_a(w^\ell, Z^\ell) = D_{(y^\ell, Z^{\ell-1})}^\beta f_a(w^\ell, y^\ell, Z^{\ell-1})\big|_{y^\ell = y_a^\ell(w^\ell) + Y^\ell}.$$

Thus $D_{Z^\ell}^\beta f_a(w^\ell, 0) = D_{(y^\ell, Z^{\ell-1})}^\beta f_a\big(w^\ell, y_a^\ell(w^\ell), 0\big)$ is the formal Taylor expansion at a of the regular function on N_ℓ given by the restriction of $D_{y^\ell}^\delta f_\gamma(w^\ell, y^\ell)$ to N_ℓ.

Lemma 9.12. *Let $f(w^\ell, Z^\ell) \in \widehat{\mathscr{O}}_{M,a_0} = \underline{k}[[w^\ell, Z^\ell]]$. Then we can write f uniquely as*

$$f(w^\ell, Z^\ell) = \sum_{i=1}^{s_\ell} q_i(w^\ell, Z^\ell) f_{i,a_0}^{\ell-1}(w^\ell, Z^\ell) + r(w^\ell, Z^\ell) \mod (Z^\ell)^{K+1},$$

where

$$q_i(w^\ell, Z^\ell) = \sum_{\substack{\beta \in \Box_i^\ell \\ |\beta| \le K - d_i}} c_{i\beta}(w^\ell)(Z^\ell)^\beta , \qquad i = 1, \dots, s_\ell ,$$

$$r(w^\ell, Z^\ell) = \sum_{\substack{\gamma \in \mathbb{N}^{r_\ell} \setminus \mathfrak{N}^\ell \\ |\gamma| \le K}} c_\gamma(w^\ell)(Z^\ell)^\gamma .$$

Moreover, if $f(w^\ell, Z^\ell) = \sum_{\gamma \in \mathbb{N}^{r_\ell}} f_\gamma(w^\ell)(Z^\ell)^\gamma$, *where each* f_γ *is induced by an element of* $\mathcal{O}(U)_{V'_\ell}$, *then there is a Zariski-open neighbourhood* V_ℓ *of* a_0 *in* V'_ℓ *in which each* $c_{i\beta}$ *or* c_γ *is the restriction to* N_ℓ *of an element of* $\mathcal{O}(U)_{V_\ell}$.

Proof. Let $A(w^\ell, Z^\ell)$ denote the square matrix with entries in $\underline{k}[[w^\ell, Z^\ell]]$ whose columns are the partial derivatives of order $\le K$ with respect to Z^ℓ of

$$\text{(9.13)} \qquad \begin{array}{ll} (Z^\ell)^\beta f_{i,a_0}^{\ell-1}(w^\ell, Z^\ell) , & \beta \in \Box_i^\ell , \quad |\beta| \le K - d_i , \quad i = 1, \dots, s_\ell , \\ (Z^\ell)^\gamma , & \gamma \in \mathbb{N}^{r_\ell} \setminus \mathfrak{N}^\ell , \quad |\gamma| \le K . \end{array}$$

The rows of A are indexed by $\gamma \in \mathbb{N}^{r_\ell}$, $|\gamma| \le K$, and the columns are indexed by (i, β) and γ as in (9.13). To specify A precisely, let us say that the rows are listed by $\gamma \in \mathfrak{N}^\ell$ in ascending order followed by $\gamma \notin \mathfrak{N}^\ell$ in ascending order, and that the columns are listed first by (i, β) in ascending order of $\alpha^i + \beta$, followed by γ in ascending order.

Since $\exp f_{i,a_0}^{\ell-1} = \alpha^i$ for all i, it follows that $A(0,0)$ is lower triangular with 1's on the diagonal. In particular, $\det A(0,0) \ne 0$. Therefore $A(w^\ell, 0)$ is invertible.

Given $f \in \underline{k}[[w^\ell, Z^\ell]]$, let $F(w^\ell)$ be the (column) vector with entries $(D^\gamma_{Z^\ell} f)(w^\ell, 0)$, $\gamma \in \mathbb{N}^{r_\ell}$, $|\gamma| \le K$, ordered in the same way as the rows of A. Then there is a unique (column) vector $C(w^\ell)$ (with entries $c_{i\beta}(w^\ell)$, $c_\gamma(w^\ell)$ listed in the same way as the columns of A) such that $F(w^\ell) = A(w^\ell, 0) \cdot C(w^\ell)$; this *is* the first assertion of the lemma.

Each entry of $A(w^\ell, 0)$ is (the Taylor expansion at a_0 of) the restriction to N_ℓ of an element of $\mathcal{O}(U)_{V'_\ell}$ (by 9.11). The second assertion then follows from Cramer's rule. $\square$

Now, for each $i > s_\ell$, we apply Remark 9.11 and Lemma 9.12 to $f_i^{\ell-1}$:

$$f_i^{\ell-1}(w^\ell, Z^\ell) = \sum_{j=1}^{s_\ell} q_j^\ell(w^\ell, Z^\ell) f_j^{\ell-1}(w^\ell, Z^\ell) + r^\ell(w^\ell, Z^\ell) \mod (Z^\ell)^{K+1} .$$

Definition 9.14. *For* (3_ℓ), *we set*

$$f_i^\ell(w^\ell, Z^\ell) = f_i^{\ell-1}(w^\ell, Z^\ell) , \qquad i = 1, \dots, s_\ell ,$$

$$f_i^\ell(w^\ell, Z^\ell) = f_i^{\ell-1}(w^\ell, Z^\ell) - \sum_{j=1}^{s_\ell} q_j^\ell(w^\ell, Z^\ell) f_j^{\ell-1}(w^\ell, Z^\ell) , \quad i = s_\ell + 1, \dots, s .$$

Properties (a_ℓ), (c_ℓ) and (d_ℓ) follow. This completes the construction of (1_ℓ)–(3_ℓ).

To complete the proof of 9.6: Take $N = N_p$ and let $f_i = \sum c_{i\gamma} Z^\gamma$ denote $f_i^p = \sum c_{i\gamma}^p (Z^p)^\gamma$, $i = 1, \ldots, s$. (Note that $w = w^p$ and $Z = Z^p$.) We must find a Zariski-open neighbourhood V of a_0 in V_p, with respect to which N and the f_i satisfy the conditions of 9.6. Set $S_{(f)} = \{a \in |N| : \mu(f_{i,a}) \geq d_i,\ i = 1, \ldots, s\}$. Then $S_{(f)} = \{a \in |N| : \mu_a(c_{i\gamma}) \geq d_i - |\gamma|, |\gamma| < d_i,\ i = 1, \ldots, s\}$, and properties (4), (5) of (7.2) are satisfied by the $f_{i,a}$, $a \in S_{(f)}$.

At a_0, properties (1) and (2) of (7.2) are consequences of the fact that $\exp f_{i,a_0} = \alpha^i$, $i = 1, \ldots, s$. The property that $\mu(f_{i,a}) \leq d_i$, $i = 1, \ldots, s$, is open in N with respect to the Zariski topology. Likewise, for each ℓ and $d \in \mathbb{N}$, property $(2_{\ell,d})$ of (7.2) is open. Thus there is a Zariski-open neighbourhood V' of a_0 in V_p such that, for all $a \in N \cap V'$, $\mu(f_{i,a}) \leq d_i$, $i = 1, \ldots, s$, and such that $(2_{\ell,d})$ is satisfied on $S_{(f)} \cap V'$ for all ℓ and each $d \leq k\big(\mathfrak{N}(\ell)\big)$. (Recall Definition 8.3.) By the Stabilization Theorem 8.1, property (2) of (7.2) is satisfied throughout $S_{(f)} \cap V'$. (This is the one place where we use 8.1.)

Consider the ideals $I_{S_{(f)}}^k$ and I_S^k of Theorem 7.14 (at a_0), where $k \geq \max d_i - 1$. $I_{S_{(f)}}^k$ is generated by the ideal of N and the $D^\beta c_{i\gamma}$, $|\beta| \leq d_i - |\gamma|$; also, I_S^k is generated by explicitly determined elements of $O(U)$ (Remarks 9.1 and Definition 7.13). There is a Zariski-open neighbourhood of a_0 in U, in which $H_{X,\cdot} \leq H_{X,a_0}$. Since $\mathcal{O}_M$ is coherent, by Theorem 7.14 and Remarks 9.5 there is a Zariski-open neighbourhood V of a_0 in V' such that $S_{(f)} \cap V = S_{H_{X,a_0}} \cap V = \{a \in V :\ H_{X,a} = H_{\mathfrak{N}}\}$. Property (3) of (7.2) holds at all such a, by Lemma 7.5. This completes the proof of Theorem 9.6. $\square$

Remarks 9.15. (1) r in Theorem 9.6 is *not* determined by the Hilbert-Samuel function H_{X,a_0}. For example, consider the following diagrams $\mathfrak{N}_1, \mathfrak{N}_2 \in \mathscr{D}(3)$: $\mathfrak{N}_1 = \mathbb{N} \times \mathfrak{N}_1^*$, where $\mathfrak{N}_1^* \subset \mathbb{N}^2$ has vertices $\{\beta \in \mathbb{N}^2 : |\beta| = 3\}$, and $\mathfrak{N}_2 = (\mathbb{N} \times \mathfrak{N}_2^*)\backslash\{(0,0,2)\}$, where the vertices of $\mathfrak{N}_2^* \subset \mathbb{N}^2$ are $(0,2)$, $(2,1)$ and $(4,0)$. Then $\mathfrak{N}_1$, $\mathfrak{N}_2$ are both monotone, $H_{\mathfrak{N}_1} = H_{\mathfrak{N}_2}$ (in the language of Sect. 7), but for $\mathfrak{N}_2$ all 3 variables are essential.

(2) Write $e = e_{X,a_0}$, where $e_{X,a_0} := H_{X,a_0}(1) - 1$. Then $n - r \leq e$ since $\mathfrak{N}$ has $n - \big(H_{\mathfrak{N}}(1) - 1\big) = n - e$ vertices of order 1, each representing an essential variable.

(3) Consider a sequence of transformations (6.7) whose centres are $1/2$-admissible. Suppose that $\big(N_p(a), \mathscr{F}_1(a), \mathscr{E}_1(a) = E(a)\backslash E^1(a)\big)$ and $\big(N_q(a), \mathscr{G}_1(a), \mathscr{E}_1(a)\big)$ are two presentations (perhaps merely formal) of $H_{X_j,\cdot}$ at $a \in M_j$, of codimensions $1 \leq p < q$, respectively. The construction of Ch. II (applied with $\big(N_p(a), \mathscr{F}_1(a), \mathscr{E}_1(a)\big)$ playing the role of "$\big(N_1(a), \mathscr{H}_1(a), \mathscr{E}_1(a)\big)$" in 6.12) gives $\mathrm{inv}_{q-p+1}(a) = (H_{X_j,a}, s_1(a); 1, 0; \ldots;\ 1, 0)$ (i.e., (1,0) is listed $q - p$ times) and, moreover, provides an (equivalent) codimension q presentation $\big(N_q'(a), \mathscr{G}_1'(a), \mathscr{E}_1(a)\big)$ of $H_{X_j,\cdot}$ at a, where $N_q'(a) \subset N_p(a)$ (cf. Remark 6.16). (If we begin with a codimension $p = 0$ presentation of $H_{X_j,\cdot}$ (e.g., a standard basis), then Construction 4.18 provides an equivalent codimension $p = 1$ presentation as used in the preceding statement.) In general, therefore, we modify the constructive definition of inv_X in the following way: If $\big(N_r(a), \mathscr{C}_1(a), \mathscr{E}_1(a) =$

$E(a)\backslash E^1(a)\big)$ is a codimension r presentation of $\mathrm{inv}_{1/2} = H_{X_j,\cdot}$ at $a \in M_j$, where $n-r < e := e_{X_j,a}$, then we put $\mathrm{inv}_{e-(n-r)}(a) = (H_{X_j,a}, s_1(a); 1,0; \ldots; 1,0)$ (i.e., $(1,0)$ is inserted $e-(n-r)-1$ times). (If $n-r = e$ (i.e., X_j is smooth at a), we just have $\mathrm{inv}_1(a) = \big(H_{X_j,a}, s_1(a)\big)$.) As in Sect. 6, $\big(N_r(a), \mathscr{H}_1(a), \mathscr{E}_1(a)\big)$, where $\mathscr{H}_1(a) = \mathscr{C}_1(a) \cup (E^1(a), 1)$, is a codimension r presentation of inv_1 at a. If $n-r < e$, therefore, $\big(N_r(a), \mathscr{H}_1(a), \mathscr{E}_1(a)\big)$ is a codimension r presentation of $\mathrm{inv}_{e-(n-r)}$ at a, so we re-index it as $\big(N_r(a), \mathscr{H}_{e-(n-r)}(a), \mathscr{E}_{e-(n-r)}(a)\big)$ (also setting $E^q(a) = \emptyset$, $1 < q \le e-(n-r)\big)$. Then the following term $\nu_{e-(n-r)+1}(a)$ of $\mathrm{inv}_X(a)$ (or $\nu_2(a)$ in the case that $n-r = e$) is given by (the analogue of) 6.12, and the definition of inv_X proceeds as in Sect. 6. The resulting definition of inv_X does not depend on r (nor on the ambient dimension n; cf. Remarks 13.1) and it agrees with that of Sect. 6 in the hypersurface case.

Chapter IV. Desingularization theorems

Theorem 1.14 is used in this chapter to obtain several desingularization theorems. Let $\mathscr{A}$ denote any of the classes of local-ringed spaces $X = (|X|, \mathscr{O}_X)$ over $\underline{k}$ in (0.2) (1)–(3). We prove embedded resolution of singularities in Sect. 10 for geometric spaces $X \in \mathscr{A}$. (cf. Remarks 1.7(2).) In the algebraic and analytic categories of (0.2) (1),(2), algebraic techniques make it possible to prove resolution of singularities under more general hypotheses on X (Sect. 11); for example, for spaces X that are not necessarily reduced. We recover, in particular, the theorems of Hironaka [H1]. Our desingularization algorithm does not, *a priori*, exclude the possibility of blowing up "resolved points"; i.e., a prescribed centre of blowing up may include points where X_j is smooth and has only normal crossings with respect to E_j. (See Example 2.3.) In Sect. 12, we show how to modify our invariant to avoid blowing up resolved points.

The desingularization theorems of Sects. 10–12 are stated for spaces X such that X is globally embedded in a smooth ambient space M and $|X|$ is quasi-compact (so that inv_X has a maximal locus which provides a smooth centre of blowing up). These hypotheses are relaxed in Sect. 13. We deduce universal "embedded" resolution of singularities for spaces X that are not necessarily globally embedded. For real or complex analytic spaces that are not necessarily compact, we prove canonical resolution of singularities by locally finite sequences of blowings-up with global smooth centres. (For example, we recover Hironaka's theorem on complex analytic spaces [AHV1,2], [H2].)

10. A geometric desingularization algorithm

Let $M \in \mathscr{A}$ be a manifold, and let $X = (|X|, \mathscr{O}_X)$ denote a closed subspace of M. Recall that $\mathrm{Reg}\, X \subset |X|$ denotes the set of smooth points of X, and $\mathrm{Sing}\, X := |X| \backslash \mathrm{Reg}\, X$. Clearly, if X is a hypersurface, then $\mathrm{Sing}\, X$ is Zariski-closed in $|X|$.

Proposition 10.1. *Suppose $\mathcal{A}$ is one of the classes of (0.2) (1),(2) (so $\mathcal{A}$ satisfies (3.9)). Assume $X \neq \emptyset$. Then there is a proper closed subspace Y of X such that $|Y| = \operatorname{Sing} X$.*

Proof. Assume first that X is a subspace of a manifold $M \in \mathcal{A}$ of pure dimension n. Let $\mathcal{I}_X$ be the ideal of X in $\mathcal{O}_M$. If $a \in |X|$, define $e_{X,a} := H_{X,a}(1) - 1$ (the local embedding dimension of X at a). For each $e \in \mathbb{N}$, let $\mathcal{J}(e) \subset \mathcal{O}_M$ denote the ideal of finite type generated locally by $\mathcal{I}_X$ and the minors of order $n-e+1$ of the Jacobian matrix (with respect to regular local coordinates) of a system of local generators of $\mathcal{I}_X$ (cf. 7.11). Thus $\mathcal{I}_X \subset \mathcal{J}(e)$ and $\operatorname{supp} \mathcal{O}_M/\mathcal{J}(e) = \{x \in |X| : \ e_{X,x} \geq e\}$. Define $\mathcal{T}(e) := [\mathcal{I}_X : \ \mathcal{J}(e)] \supset \mathcal{I}_X$. Thus $\mathcal{I}_{X,a} = \mathcal{J}(e)_a$ if and only if $\mathcal{T}(e)_a = \mathcal{O}_{M,a}$. Then:

(10.2) $a \in \operatorname{Reg} X$ if and only if $\mathcal{I}_{X,a} = \mathcal{J}(e_{X,a})_a$.

To prove (10.2): Without loss of generality, we can assume that $e_{X,a} = n$ (by passing to a local embedding of X in a submanifold of dimension $e_{X,a}$). Then $\mathcal{J}(e_{X,a})_a \subset \mathcal{O}_{M,a}$ is the ideal generated by $\mathcal{I}_{X,a}$ and the partial derivatives (with respect to regular local coordinates) $\partial f/\partial x_i$, for all $f \in \mathcal{I}_{X,a}$. (Since $e_{X,a} = n$, each $(\partial f/\partial x_i)(a) = 0$.) We have to show that $\mathcal{I}_{X,a} = \mathcal{J}(n)_a$ if and only if $\mathcal{I}_{X,a} = 0$. "If" is trivial. Conversely, if $\mathcal{I}_{X,a} \neq 0$, then the order of $\mathcal{J}(n)_a$ is strictly less than that of $\mathcal{I}_{X,a}$.

Now let $e_* = \min_{a \in |X|} e_{X,a}$ and let $Y \subset X$ denote the subspace defined by the sheaf of ideals $\mathcal{I}_Y = \bigcap_{e \geq e_*} \big(\mathcal{T}(e) + \mathcal{J}(e)\big)$.

Let $a \in |X|$. Since $\mathcal{J}(e)_a = \mathcal{O}_{M,a}$ if and only if $e > e_{X,a}$, it follows that $\mathcal{I}_{Y,a} = \mathcal{O}_{M,a}$ if and only if $\mathcal{T}(e)_a = \mathcal{O}_{M,a}$, $e_* \leq e \leq e_{X,a}$. Therefore, $\mathcal{I}_{Y,a} = \mathcal{O}_{M,a}$ if and only if $\mathcal{I}_{X,a} = \mathcal{J}(e)_a$, $e_* \leq e \leq e_{X,a}$. On the other hand, if $e_1 \geq e_2$, then $\mathcal{J}(e_1) \supset \mathcal{J}(e_2)$. Therefore, $\mathcal{I}_{Y,a} = \mathcal{O}_{M,a}$ if and only if $\mathcal{I}_{X,a} = \mathcal{J}(e_{X,a})_a$. By (10.2), $a \in \operatorname{Reg} X$ if and only if $a \notin |Y|$; i.e., $|Y| = \operatorname{Sing} X$.

Of course, $\mathcal{I}_Y \supset \mathcal{I}_X$. Consider $a \in |X|$ such that $e_{X,a} = e_*$. If $a \in \operatorname{Reg} X$, then $\mathcal{I}_{Y,a} = \mathcal{O}_{M,a} \underset{\neq}{\supsetneq} \mathcal{I}_{X,a}$. If $a \notin \operatorname{Reg} X$, then $\mathcal{I}_{X,a} \underset{\neq}{\subsetneq} \mathcal{J}(e_{X,a})_a$, so that $\mathcal{I}_{Y,a} = \mathcal{T}(e_{X,a})_a + \mathcal{J}(e_{X,a})_a \underset{\neq}{\supsetneq} \mathcal{I}_{X,a}$.

In this proof, it is clear that the assumption that X is globally embedded in M is only a matter of convenience; the arguments can be rewritten without this assumption. $\square$

Consider a sequence of transformations (1.1) with inv_X-admissible centres. If $a \in |M_j|$, set $S_{\operatorname{inv}_X}(a) = \{x \in |M_j| : \ \operatorname{inv}_X(x) = \operatorname{inv}_X(a)\}$.

Remarks 10.3. (1) If $a \in \operatorname{Sing} X_j$, then $S_{\operatorname{inv}_X}(a) \subset \operatorname{Sing} X_j$ because the Hilbert-Samuel function $H_{X_j,x}$ already distinguishes between smooth and singular points x of X_j.

(2) Suppose that X_j is smooth. Let $S_j := \{x \in |X_j| : \ s_1(x) > 0\}$; then S_j is a Zariski-closed subset of $|X_j|$, by Proposition 6.6. Clearly, if $a \in S_j$, then $S_{\operatorname{inv}_X}(a) \subset S_j$.

A desingularization algorithm; proof of Theorem 1.6. Suppose $|X|$ is quasi-compact, so that inv_X takes only finitely many values on $|X|$. We can get an

inv_X-admissible sequence (1.1) by defining the centres of blowing up C_j recursively as follows: Assume that $\sigma_1, \ldots, \sigma_j$ have been defined. We introduce the extended invariant $\mathrm{inv}_X^e(a)$, $a \in M_j$, as in Remark 1.16, using any total ordering on the subsets of E_j. If X_j is not smooth, let C_j denote the locus of maximal values of inv_X^e on $\mathrm{Sing}\, X_j$. Since $\mathrm{Sing}\, X_j$ is Zariski-closed, it follows from Theorem 1.14 and Remarks 10.3 (1) that C_j is smooth. By 1.14 (4), after finitely many blowings-up with such centres, X_j is smooth.

If X_j is smooth, let C_j denote the locus of maximal values of inv_X^e on S_j. By Theorem 1.14 and 10.3 (2), C_j is smooth. Therefore, X_{j+1} is smooth. After finitely many blowings up $\sigma_{j+1}, \ldots, \sigma_k$ with such centres, $S_k = \emptyset$. It is clear from the definition of s_1 that, if X_k is smooth and $S_k = \emptyset$, then each $H \in E_k$ which intersects X_k is the strict transform in M_k of $\sigma_{i+1}^{-1}(C_i)$, for some i such that X_i is smooth along C_i; therefore, X_k and E_k simultaneously have only normal crossings. We have proved Theorem 1.6.

Remark 10.4 It may happen in Theorem 1.6 that $X' = \emptyset$; 1.6 is a meaningful geometric desingularization theorem at least in the case that $\mathrm{Reg}\, X$ is Zariski-dense in $|X|$.

Geometric spaces. Let $\mathscr{A}$ be any of the classes of (0.2) (1), (2). Suppose $X \in \mathscr{A}$. Let S be a subset of $|X|$. We say that a subspace Y of X (in $\mathscr{A}$) is *a smallest subspace whose support contains* S provided that $S \subset |Y|$ and if $T \subset Y$, $S \subset |T|$ then $T = Y$. Since X is locally Noetherian, there is a *unique* smallest subspace $\overline{S}$ of X whose support contains S; $\overline{S}$ is the intersection of all subspaces of X (in $\mathscr{A}$) whose support contains S. In particular, there is a unique smallest subspace X_* of X such that $|X_*| = |X|$; namely, $X_* = \overline{|X|}$.

Definition 10.5. *We say that X is* **geometrically reduced** *if $X = X_*$. We say that X is a* **geometric space** *if $\mathrm{Reg}\, X$ is Zariski-dense in X (i.e., $X = \overline{\mathrm{Reg}\, X}$).*

Proposition 10.6. *X is geometric if and only if X is geometrically reduced.*

Proof. If X is geometric, then of course X is geometrically reduced. Assume X is geometrically reduced. By Proposition 10.1, there is a subspace X_1 of X such that $X_1 \subsetneqq X$ and $|X| \backslash \mathrm{Reg}\, X = |X_1|$. Let $X_0 = \overline{\mathrm{Reg}\, X}$, so that X_0 is geometrically reduced. Define $X_2 := \overline{|X| \backslash |X_0|}$. Then X_2 is geometrically reduced. Clearly, $X_2 \subset X_1$ and $|X| = |X_0| \cup |X_2|$. Since X is geometrically reduced, $X \subset X_0 \coprod X_2$, where $X_0 \coprod X_2$ is the subspace of M (with support $|X_0| \cup |X_2|$) defined by the ideal of finite type $\mathscr{I}_{X_0} \cdot \mathscr{I}_{X_2}$. Moreover, $\mathrm{Reg}\, X_2 \subset |X_0|$ because a smooth point of X_2 outside $|X_0|$ would necessarily be a smooth point of $X_0 \coprod X_2$, hence of X, in contradiction to the definition of X_0.

We must show $X_2 = \emptyset$. Set $Y = X_2$, and let Y_0, Y_1, Y_2 be the analogues for Y of X_0, X_1, X_2 above. It is enough to show $Y = Y_2$ because then, if $Y \neq \emptyset$, $Y = Y_2 \subset Y_1 \subsetneqq Y$; a contradiction. Now, since $\mathrm{Reg}\, Y \subset |X_0|$, $|Y_0| \subset |X_0|$, so that $|Y| \backslash |Y_0| \supset |Y| \backslash |X_0| = |X| \backslash |X_0|$. Therefore, $Y \supset \overline{|Y| \backslash |Y_0|} \supset \overline{|X| \backslash |X_0|} = Y$; i.e., $Y = Y_2$. $\square$

The desingularization algorithm above can be modified so that a quasi-compact geometric space is desingularized by transformations that preserve the class of geometric spaces: Let $M \in \mathscr{A}$ be a manifold and X a closed subspace of M. Assume X is geometrically reduced. We consider a sequence of transformations (1.1) where each X_{j+1} is defined not as the strict transform of X_j, but rather as the (unique) smallest subspace of $\sigma_{j+1}^{-1}(X_j)$ whose support contains $|\sigma_{j+1}^{-1}(X_j)|\backslash|\sigma_{j+1}^{-1}(C_j)|$; in this case, we say that X_{j+1} is the *geometric strict transform* of X_j by σ_{j+1}. Each X_{j+1} is geometrically reduced.

Using the geometric strict transform, our invariant $\mathrm{inv}_X(a)$, $a \in M_j$, $j = 0, 1, 2, \ldots$, can be defined by induction on j as before provided that the centres C_i, $i < j$, are inv_X-admissible in the sense of (1.2), and our desingularization algorithm makes sense exactly as before for the following reason: Given j, let Y_{j+1} and X_{j+1} denote the strict and geometric strict transforms of X_j, respectively. Let $a \in X_j$ and $a' \in \sigma_{j+1}^{-1}(a)$. Since $H_{X_j,\cdot}$ is locally constant on C_j and $\mathscr{I}_{X_{j+1},a'} \supset \mathscr{I}_{Y_{j+1},a'}$, $H_{X_{j+1},a'} \le H_{Y_{j+1},a'} \le H_{X_j,a}$, and if $H_{X_{j+1},a'} = H_{X_j,a}$, then $X_{j+1,a'} = Y_{j+1,a'}$. We get the following variant of 1.6:

Theorem 10.7. *Suppose that X is geometrically reduced and that $|X|$ is quasi-compact. Then there is a finite sequence of blowings-up (1.1) with smooth* inv_X*-admissible centres C_j (where each X_{j+1} denotes the geometric strict transform of X_j) such that:*

(1) For each j, either $C_j \subset \mathrm{Sing}\, X_j$ or X_j is smooth and $C_j \subset X_j \cap E_j$.

(2) Let X' and E' denote the final geometric strict transform and exceptional set, respectively. Then X' is smooth and X', E' simultaneously have only normal crossings.

If σ denotes the composite of the sequence of blowings-up σ_j, then $\sigma(E') \subset \mathrm{Sing}\, X$; clearly, $\sigma^{-1}(\mathrm{Reg}\, X)$ is a smooth open subset of $|X'|$, and $\overline{\sigma^{-1}(\mathrm{Reg}\, X)}$ is open and closed in X'. (In fact, if $T = \overline{\sigma^{-1}(\mathrm{Reg}\, X)}$ and $\mathscr{J}_T \subset \mathscr{O}_{X'}$ denotes the ideal of T, then $|T| = |X'|\backslash \mathrm{supp}\, \mathscr{J}_T$.)

11. Algebraic desingularization theorems

In this section, $\mathscr{A}$ denotes any of the categories in (0.2) (1) or (2), so that $\mathscr{A}$ satisfies (3.9). Let $X = (|X|, \mathscr{O}_X) \in \mathscr{A}$. Assume that X is a closed subspace of a manifold $M = (|M|, \mathscr{O}_M) \in \mathscr{A}$, and let $\mathscr{I}_X \subset \mathscr{O}_M$ be the ideal sheaf of X. Since $\mathscr{O}_M$ is a coherent sheaf of rings and $\mathscr{I}_X$ an ideal of finite type, the radical $\sqrt{\mathscr{I}_X} \subset \mathscr{O}_M$ is an ideal of finite type. Let X_{red} denote the subspace of M defined by the ideal sheaf $\mathscr{I}_{X_{\mathrm{red}}} := \sqrt{\mathscr{I}_X}$; thus $X_{\mathrm{red}} \subset X$ and $|X_{\mathrm{red}}| = |X|$. (X_{red} is the "associated reduced subspace" of X.)

Let $a \in |X|$. Let $\mathscr{I}_{|X|,a} \subset \mathscr{O}_{M,a}$ denote the ideal $\{f \in \mathscr{O}_{M,a} \colon \gamma^*(f) = 0$ for every homomorphism $\gamma^* \colon \mathscr{O}_{M,a} \to \underline{k}[[t]]$ such that $\mathrm{Ker}\, \gamma^* \supset \mathscr{I}_{X,a}\}$. ($\mathscr{I}_{|X|,a}$ is the ideal of germs of regular functions at a which "vanish on every formal curve" $\gamma(t)$). Thus $\mathscr{I}_{X,a} \subset \mathscr{I}_{X_{\mathrm{red}},a} \subset \mathscr{I}_{|X|,a}$. Using Artin's approximation theorems (in the case of curves; i.e., with a single independent variable t), we can replace the formal curves γ in the preceding definition by formal curves γ such that $\dim \mathcal{O}_{M,a}/\mathrm{Ker}\gamma^* = 1$ (i.e., $\mathrm{Ker}\gamma^*$ defines a regular curve which admits a formal parametrization as above, and therefore, for example in the analytic case, a convergent parametrization.) Let $V(\mathcal{I}_{|X|,a})$ be the germ of a subspace of M with associated ideal $\mathcal{I}_{|X|,a}$. It follows that $V(\mathcal{I}_{|X|,a})$ depends only on the germ $|X|_a$ of $|X|$ at a, and $V(\mathcal{I}_{|X|,a})$ is the smallest germ of a subspace with support $|V(\mathcal{I}_{|X|,a})|$. In particular, if T is (locally) a Zariski-closed subset of $|X|$, we likewise have an ideal $\mathcal{I}_{T,a} \supset \mathcal{I}_{|X|,a}$. (If $\underline{k}$ is not algebraically closed, $\mathcal{I}_{|X|}$ is not necessarily a sheaf of ideals of finite type. In the definition above, we have assumed that $\underline{k}$ is the residue field $\mathcal{O}_{M,a}/\underline{m}_{M,a}$ (e.g., as in the categories of real analytic spaces or real algebraic varieties); in general, $\mathbb{F}_a = \mathcal{O}_{M,a}/\underline{m}_{M,a}$ is an extension of the ground field $\underline{k}$, and $\underline{k}[[t]]$ would be replaced by $\mathbb{F}[[t]]$, where $\mathbb{F}$ runs over all finite extensions of $\mathbb{F}_a$.)

In the case of analytic spaces, $\mathcal{I}_{|X|,a} \subset \mathcal{O}_{M,a}$ is the ideal of elements which vanish on $|X|_a$, and $|V(\mathcal{I}_{|X|,a})| = |X|_a$. (These assertions are consequences of the curve-selection lemma, which can be proved, for example, using Sect. 10: By Theorem 1.6, $|X|$ is an image of a manifold by a proper regular mapping; therefore, any $b \in |X|\backslash\{a\}$ close enough to a can be joined to a by a "convergent curve" γ as above.) The preceding assertions for analytic spaces are not valid in the real algebraic example following.

Example 11.1. Let $X = V(x_3^2 - x_1x_2^2) \subset \mathbb{R}^3$. Then $X_{\mathrm{red}} = X$. But if $a = (a_1, 0, 0)$, where $a_1 < 0$, then $\mathcal{I}_{|X|,a} = (x_2, x_3)$.

Singular subsets of X. If $a \in |X|$, we let $S = S_{H_{X,a}}$ denote the Hilbert-Samuel subspace of a; i.e., $\mathcal{I}_S = \sum_{k=0}^{\infty} \overset{\circ}{\mathcal{I}}{}^k_S(r_k)$, where each $r_k = q - H_{X,a}(k)$ (in the notation of Remarks 9.1). In particular, $|S| = \{x \in |X|\colon H_{X,x} \geq H_{X,a}\}$.

Definitions 11.2. $\mathrm{Sing}|X| := \{a \in |X|$*:* $V(\mathcal{I}_{|X|,a})$ *is not smooth; i.e.,* $\mathcal{O}_{M,a}/\mathcal{I}_{|X|,a}$ *is not a regular local ring* $\}$.

$\mathrm{Sing}_{\dim} X := \{a \in |X|$*:* $\dim V(\mathcal{I}_{|X|,a}) < \dim X_a\}$. *(*$\dim$ *denotes the Krull dimension of the corresponding local ring; thus,* $\dim V(\mathcal{I}_{|X|,a}) = \dim \mathcal{O}_{M,a}/\mathcal{I}_{|X|,a}$.*)*

$\mathrm{Sing}_H X := \{a \in |X| :\ \mathcal{I}_{|X|,a} \subsetneqq \mathcal{I}_{|S|,a}$, *where* $S = S_{H_{X,a}}\}$.

$\Sigma := \mathrm{Sing}_H X \cup \mathrm{Sing}_{\dim} X$.

Remarks 11.3. $\mathrm{Sing}_{\dim} X \subset \mathrm{Sing}\, X_{\mathrm{red}}$ because, if $X_{\mathrm{red},a}$ is smooth, then $\mathcal{I}_{|X|,a} = \mathcal{I}_{X_{\mathrm{red}},a}$, so that $\dim V(\mathcal{I}_{|X|,a}) = \dim X_{\mathrm{red},a} = \dim X_a$. Clearly, $\mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}} \subset \mathrm{Sing}\, X$, with equality if $X = X_{\mathrm{red}}$. It is not true, in general, that $\Sigma \subset \mathrm{Sing}\, X_{\mathrm{red}}$ (or even $\mathrm{Sing}\, |X| \cup \mathrm{Sing}_{\dim} X$) because $H_{X,x}$ need not be locally constant at a smooth point of X_{red}. (For example, consider the complex analytic subspace X of $\mathbb{C}^2$ defined by the intersection of ideals $\mathcal{I}_X = (x^2) \cap (x^4, y)$. Then $\mathcal{I}_{X_{\mathrm{red}}} = (x)$ but $\mathrm{Sing}_H X = \{0\}$.)

Note that $\mathrm{Sing}|X|$ (or even $\mathrm{Sing}\, |X| \cup \mathrm{Sing}_{\dim} X$) need not be a Zariski-closed subset of $|X|$; for example, the real algebraic subset $X\colon z^3 - x^2yz - x^4 = 0$ of $\mathbb{R}^3$ is smooth except on the half-line $x = z = 0$, $y \geq 0$.

Theorem 11.4. *Let $a \in |X|$ and let $S = S_{H_{X,a}}$. Then:*

(1) $e_{S,a} \leq \dim X_a$; moreover, $S_a = X_a$ if and only if X_a is smooth.

(2) Suppose that $a \notin \Sigma$. Then $X_{\mathrm{red},a}$ is smooth, $S_a = X_{\mathrm{red},a}$, and $\mathrm{inv}_X(a) = (H_{X,a}, 0; \infty)$. (Here inv_X is "at year zero".)

(Recall that, for all $a \in |X|$, $e_{X,a} := H_{X,a}(1) - 1$ is the minimal local embedding dimension of X at a.) We will need the following three lemmas.

Lemma 11.5. *Let A be a Noetherian local ring, and let $I \subset \mathfrak{q}$ be ideals of A, where $\mathfrak{q}$ is prime. Write $\sqrt{I} = \bigcap \mathfrak{p}_i$ for the (unique) irredundant prime decomposition of $\sqrt{I}$. If $\dim A/\mathfrak{q} = \dim A/I$, then $\mathfrak{q} = \mathfrak{p}_i$ for some i.*

Proof. Since $\mathfrak{q} \supset I$ and $\mathfrak{q}$ is prime, $\mathfrak{q} \supset \sqrt{I} = \bigcap \mathfrak{p}_i$; therefore, $\mathfrak{q} \supset \mathfrak{p}_i$ for some i, say $i = 1$. Then $\dim A/\mathfrak{p}_1 \geq \dim A/\mathfrak{q} = \dim A/\sqrt{I} = \max \dim A/\mathfrak{p}_i$, so that $\dim A/\mathfrak{p}_1 = \dim A/\mathfrak{q}$. Therefore, $\mathfrak{q} = \mathfrak{p}_1$. □

Example 11.6. Let X be the real-analytic space $V(x_3^3 - x_1^2 x_2^3) \subset \mathbb{R}^3$ and let $a = (a_1, 0, 0)$, $a_1 \neq 0$. Then $\mathscr{I}_{|X|,a}$ is generated by $x_3 - x_1^{2/3} x_2$; $V(\mathscr{I}_{|X|,a})$ is a component of $X_{\mathrm{red},a} = X_a$, by 11.5. But there is another component, given by $x_3^2 + x_1^{2/3} x_2 x_3 + x_1^{4/3} x_2^2$.

Lemma 11.7. *Let A be a Noetherian local ring. Let $\mathfrak{p} \subset I$ be ideals of A, where $\mathfrak{p}$ is prime and $I = \sqrt{I}$. If $\dim A/\mathfrak{p} = \dim A/I$, then $I = \mathfrak{p}$.*

Proof. Let $\bigcap_{i=1}^s = \mathfrak{p}_i$ denote the irredundant prime decomposition of I. Then $\dim A/I = \max \dim A/\mathfrak{p}_i = \dim A/\mathfrak{p}_1$ say. Now, $\mathfrak{p} \subset I \subset \mathfrak{p}_1$ and $\dim A/I = \dim A/\mathfrak{p}_1$; therefore, $\mathfrak{p} = \mathfrak{p}_1$. Thus $\mathfrak{p}_1 = \mathfrak{p} \subset \mathfrak{p}_i$, $i = 1, \ldots, s$; a contradiction unless $s = 1$. Hence $I = \mathfrak{p}$. □

Lemma 11.8. *Suppose $a \notin \Sigma$ and $V(\mathscr{I}_{|X|,a})$ is smooth. Then $V(\mathscr{I}_{|X|,a}) = X_{\mathrm{red},a}$.*

Proof. Since $V(\mathscr{I}_{|X|,a})$ is smooth and $\dim V(\mathscr{I}_{|X|,a}) = \dim X_{\mathrm{red},a}$, $V(\mathscr{I}_{|X|,a})$ is a component of $X_{\mathrm{red},a}$, by Lemma 11.5. Let $\bigcap \mathfrak{p}_i$ denote the irredundant prime decomposition of $\mathscr{I}_{X_{\mathrm{red}},a}$; thus the $X_i := V(\mathfrak{p}_i)$ are the irreducible components of $X_{\mathrm{red},a}$. Since $a \notin \mathrm{Sing}_H X$, $|X_i| = |V(\mathscr{I}_{|X|,a})|$ for all i. (Otherwise, the Hilbert-Samuel function would not be locally constant, by a simple semicontinuity argument.) Therefore, for each i, $\dim \mathscr{O}_{M,a}/\mathscr{I}_{|X|,a} \leq \dim \mathscr{O}_{M,a}/\mathfrak{p}_i \leq \dim \mathscr{O}_{M,a}/\mathscr{I}_{X_{\mathrm{red}},a} = \dim \mathscr{O}_{M,a}/\mathscr{I}_{|X|,a}$, so that all are equal; since $\mathfrak{p}_i \subset \mathscr{I}_{|X|,a}$ are both prime, it follows that $\mathfrak{p}_i = \mathscr{I}_{|X|,a}$. In particular, there is only one irreducible component X_i. □

Proof of Theorem 11.4. We identify $\widehat{\mathscr{O}}_{M,a}$ with $\underline{k}[[X]] = \underline{k}[[X_1, \ldots, X_n]]$ using local coordinates. Let $\mathfrak{N} = \mathfrak{N}(I)$ denote the diagram of initial exponents of $I = \widehat{\mathscr{I}}_{X,a} \subset \underline{k}[[X]]$. We use the notation of (7.1). Let $f_i(X) = f_i(W, Z) \in I$,

$i = 1, \ldots, s$, be the standard basis of I, so that the f_i satisfy (7.2) (1)–(5). Let $J = \widehat{\mathcal{I}_{S,a}} \subset \underline{k}[[X]]$.

(1) It follows from Theorem 7.14 and property (4) of (7.2) that $Z = (Z_1, \ldots, Z_r) \subset J$, so that $e_{S,a} \leq n - r$. On the other hand, $\dim X_a = \dim \underline{k}[[X]]/I \geq n - r$. (Consider the homogeneous ideal H in $\underline{k}[[X]]$ generated by the initial monomials $\operatorname{mon} f_i(X)$. Then $\dim X_a = \dim \underline{k}[[X]]/H$ since dim is determined by the Hilbert-Samuel function, and $\dim \underline{k}[[X]]/H \geq n-r$ because the $\operatorname{mon} f_i(X)$ are independent of $W = (W_1, \ldots, W_{n-r})$.) Thus $e_{S,a} \leq n - r \leq \dim X_a \leq e_{X,a}$, and it follows that $S_a = X_a$ if and only if X_a is smooth.

(2) Suppose that $a \notin \Sigma$; i.e., $\dim V(\mathcal{I}_{|X|,a}) = \dim X_a \geq n - r$ and $\mathcal{I}_{|S|,a} = \mathcal{I}_{|X|,a}$. Then $(Z) = (Z_1, \ldots, Z_r) \subset J = \widehat{\mathcal{I}_{S,a}} \subset \widehat{\mathcal{I}_{|S|,a}} = \widehat{\mathcal{I}_{|X|,a}}$, so that $\dim V(\mathcal{I}_{|X|,a}) = n - r$ and, by Lemma 11.7, $\widehat{\mathcal{I}_{|X|,a}} = (Z_1, \ldots, Z_r)$; in particular, $V(\mathcal{I}_{|X|,a})$ is smooth and $S_a = V(\mathcal{I}_{|X|,a})$. By Lemma 11.8, $V(\mathcal{I}_{|X|,a}) = X_{\mathrm{red},a}$. For each $i = 1, \ldots, s$, write $f_i(W, Z) = Z^{\gamma^i} + \Sigma_{\gamma \in \mathbb{N}^r} c_{i\gamma}(W) Z^{\gamma}$, where $\alpha^i = (0, \gamma^i) \in \mathbb{N}^{n-r} \times \mathbb{N}^r$; then $c_{i\gamma} = 0$ if $|\gamma| < |\gamma^i|$. (Otherwise, by Theorem 7.14, $\widehat{\mathcal{I}_{S,a}} \supsetneqq (Z)$.) It follows from the constructive definition of inv_X that $\mathrm{inv}_X(a) = (H_{X,a}, 0; \infty)$. □

Corollary 11.9. $\Sigma = \mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}}$. *If* $X = X_{\mathrm{red}}$, *then* $\Sigma = \mathrm{Sing}\, X$.

Proof. By Remarks 11.3, $\Sigma \subset \mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}}$ and, if $X = X_{\mathrm{red}}$, then $\mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}} = \mathrm{Sing}\, X$. But by Theorem 11.4, if $a \notin \Sigma$, then $a \notin \mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}}$.
□

Remarks 11.10. If $a \in |X|$, let $|S_{\mathrm{inv}_X(a)}| := \{x \in |X| : \mathrm{inv}_X(x) \geq \mathrm{inv}_X(a)\}$. (This is Zariski-closed, by 1.14 (1).) If $a \notin \Sigma$, then $\mathcal{I}_{S,a} = \mathcal{I}_{T,a}$, where $T = |S_{\mathrm{inv}_X(a)}|$, by (2) in the proof of 11.4. ($\widehat{\mathcal{I}_{S,a}} = (Z)$, as in (2), and the argument at the end shows $\widehat{\mathcal{I}_{T,a}} = (Z)$.) Define $\mathrm{Sing}_{\mathrm{inv}} X := \{a \in |X| : \mathcal{I}_{|X|,a} \subsetneqq \mathcal{I}_{T,a}\}$. It follows that $\Sigma = \mathrm{Sing}_{\mathrm{inv}} X \cup \mathrm{Sing}_{\dim} X$.

Lemma 11.11. *If $|X|$ is quasi-compact, then Σ is a Zariski-closed subset of $|X|$.*

Proof . Let $\mathrm{Sing}_H^{\mathrm{Zar}} X := \{a \in |X| :\ H_{X,\cdot}$ is not constant on any Zariski-open neighbourhood of a in $|X|\}$. Clearly, $\mathrm{Sing}_H^{\mathrm{Zar}} X$ is Zariski-closed and $\mathrm{Sing}_H X \subset \mathrm{Sing}_H^{\mathrm{Zar}} X$. Since $|X|$ is quasi-compact, $H_{X,\cdot}$ takes only finitely many values. Therefore, for all $a \in |X|$, $Z_a := \cup\{|S_{H_{X,b}}| : b \in |X|,\ H_{X,b} \not\leq H_{X,a}\}$ is Zariski-closed, and $\{x \in |X| :\ H_{X,x} \leq H_{X,a}\} = |X| \backslash Z_a$ is Zariski-open (cf. (2) $\Rightarrow$ (1) in the proof of Lemma 3.10). It follows (using Theorem 11.4 (2)) that if $a \notin \Sigma$, then $a \notin \mathrm{Sing}_H^{\mathrm{Zar}} X$; hence $\mathrm{Sing}_H^{\mathrm{Zar}} X \cup \mathrm{Sing}\, X_{\mathrm{red}} \subset \mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}} = \Sigma$. Thus $\Sigma = \mathrm{Sing}_H^{\mathrm{Zar}} X \cup \mathrm{Sing}\, X_{\mathrm{red}}$. □

Remarks 11.12. The same argument shows that, in general, any quasi-compact subset of $|X|$ admits a neighbourhood U such that $U \cap \Sigma$ is closed in the induced Zariski topology of U; in fact, such that $U \cap \Sigma = (U \cap \mathrm{Sing} X_{\mathrm{red}}) \cup \{a \in U : H_{X,\cdot}$

is not constant on any open neighbourhood of a in the induced Zariski topology of $U\}$.

Proposition 11.13. *Let $a \in \Sigma$ and let $S = S_{H_{X,a}}$. Then $|S|_a \subset \Sigma_a$.*

Proof. By restricting X to a suitable neighbourhood of a, we can assume that $H_{X,x} \le H_{X,a}$ for all $a \in |X|$, and that $H_{X,\cdot}$ takes only finitely many values, so that $\Sigma = \mathrm{Sing}_H^{\mathrm{Zar}} X \cup \mathrm{Sing}\, X_{\mathrm{red}}$, as in the proof of 11.11. Then $|S| = \{x \in |X| : H_{X,x} = H_{X,a}\}$.

First assume that $X_{\mathrm{red},a}$ is smooth; we can therefore assume that X_{red} is smooth, so that $\Sigma = \mathrm{Sing}_H X = \mathrm{Sing}_H^{\mathrm{Zar}} X$. Then $|S| \subsetneqq |X|$ since $a \in \mathrm{Sing}_H X$. Suppose the assertion is false. Then $|S|\backslash\Sigma$ is a Zariski-open subset U_1 of $|X|$ such that $U_{1,a} \ne \emptyset$ and $H_{X,x} = H_{X,a}$, $x \in U_1$. Write $|X|\backslash\Sigma = \bigcup_{j=1}^k U_j$, where the U_j are nonempty Zariski-open subsets of $|X|$ on which $H_{X,\cdot}$ takes distinct constant values. We can assume that $U_{j,a} \ne \emptyset$ for all j (by shrinking to a suitable neighbourhood of a). Even then, $k > 1$. (Otherwise, $|X| = |S| \cup \Sigma$, so there exists $b \in \Sigma\backslash|S|$. Since $H_{X,b} < H_{X,a}$, $H_{X,\cdot}$ assumes a minimal value $< H_{X,a}$. This value is attained on a Zariski-open set, in contradiction with $|X| = U_1 \cup \Sigma$.) Put $Y_1 = |X|\backslash \bigcup_{j\ge 2} U_j$, $Y_2 = |X|\backslash U_1$. Then Y_1, Y_2 are Zariski-closed subsets of $|X|$ such that $|X| = Y_1 \cup Y_2$, $a \in Y_1 \cap Y_2$, but neither $Y_1 \subset Y_2$ nor $Y_2 \subset Y_1$; this is impossible since $|X| = X_{\mathrm{red}}$ is smooth.

Secondly, assume $a \in \mathrm{Sing}\, X_{\mathrm{red}}$ but that $\mathcal{I}_{X_{\mathrm{red}},a}$ is prime. We can assume $\dim S_x \le \dim S_a$, for all $x \in |S|$ (by Zariski-semicontinuity of $H_{S,\cdot}$). Suppose there exists $b \in |S|\backslash\Sigma$. Then $S_b = X_{\mathrm{red},b}$ by 11.4, so that $\dim X_a \ge \dim S_a \ge \dim S_b = \dim X_b = \dim X_a$ (the latter equality since $H_{X,a} = H_{X,b}$); hence all are equal. Since $\mathcal{I}_{X_{\mathrm{red}},a}$ is prime, it follows from 11.7 that $\mathcal{I}_{X_{\mathrm{red}},a} = \mathcal{I}_{S_{\mathrm{red}},a}$. But, by 11.4, $e_{S_{\mathrm{red}},a} \le e_{S,a} \le \dim X_a = \dim X_{\mathrm{red},a} \le e_{X_{\mathrm{red}},a}$; therefore all are equal and $X_{\mathrm{red},a}$ is smooth. (A contradiction.)

It remains to consider the case that $\mathcal{I}_{X_{\mathrm{red}},a}$ is not prime. Let Z_i denote the distinct irreducible components of $X_{\mathrm{red},a}$, and let $X_a = \cup Y_j$ corresponding to an irredundant primary decomposition of $\mathcal{I}_{X,a}$. Then each $|Y_j| \subset |Z_i|$ for some $i = i(j)$, and each $|Z_i| = |Y_j|$ for some $j = j(i)$. For each $j = j_0$, $H_{X,a} > H_{W_{j_0},a}$, where $W_{j_0} = \bigcup_{j \ne j_0} Y_j$. It follows from semicontinuity of $H_{X,\cdot}$ that, $|S|_a \subset \cap |Y_j| \subset \cap |Z_i| \subset (\mathrm{Sing}\, X_{\mathrm{red}})_a \subset \Sigma_a$ (the latter inclusion by Theorem 11.4 (2)). $\square$

Embedded desingularization theorems. We assume that, for all $X \in \mathcal{A}$, $|X|$ is quasi-compact. (Examples include schemes of finite type over $\underline{k}$, or compact analytic spaces over $\underline{k}$.) In general, however (in view of 11.12), Theorem 11.14 below applies to desingularize X over some neighbourhood of any quasi-compact subset of $|X|$. In Sect. 13, we will obtain a global canonical desingularization theorem for non-compact analytic spaces.

Let $X \in \mathcal{A}$. Assume X is embedded in a manifold M. Set $X_0 = X$, $M_0 = M$.

Theorem 11.14 (cf. [H1, Main Theorem I*]). *There is a finite sequence of blowings-up (1.1) with smooth* inv_X*-admissible centres C_j such that:*

(1) For each j, either $C_j \subset \Sigma_j = \mathrm{Sing}_H X_j \cup \mathrm{Sing}\, X_{j,\mathrm{red}}$ or $X_{j,\mathrm{red}}$ is smooth and $C_j \subset X_j \cap E_j$.

(2) Let X' and E' denote the final strict transform of X and exceptional set, respectively. Then $H_{X',\cdot}$ is locally constant on $|X'|$, X'_{red} is smooth, and X'_{red}, E' simultaneously have only normal crossings.

Proof. The algorithm is an obvious modification of that of Theorem 1.6: (The assertions are the same when $X = X_{\mathrm{red}}$.) We define the centres of blowing up C_j as follows. Assume that $\sigma_1, \ldots, \sigma_j$ have been defined. We introduce the extended invariant $\mathrm{inv}_X^e(a)$, $a \in M_j$, as in Remarks 1.15, using any total ordering on the subsets of E_j. If $\Sigma_j \neq \emptyset$, where $\Sigma_j := \mathrm{Sing}_H X_j \cup \mathrm{Sing}_{\dim} X_j$, let C_j denote the locus of maximal values of inv_X^e on Σ_j. Since Σ_j is Zariski-closed (11.11), it follows from 1.15 and 11.13 that C_j is a smooth closed subspace of X_j. By 1.14 (4), after finitely many blowings-up with such centres, $\Sigma_j = \emptyset$; i.e. (by 11.9), $X_{j,\mathrm{red}}$ is smooth and $H_{X_j,\cdot}$ is locally constant on $|X_j|$.

Now assume $X_{j,\mathrm{red}}$ is smooth and $H_{X_j,\cdot}$ is locally constant on $|X_j|$. We consider inv_{X_j}, starting with $X_j \subset M_j$ as our spaces and E_j as our exceptional set "at year zero". (In particular, if $a \in |X_j|$, then $s_1(a) := \#\{H \in E_j : a \in H\}$.) Define $S_j := \{x \in |X_j| : s_1(x) > 0\}$ and let C_j denote the locus of maximal values of $\mathrm{inv}_{X_j}^e$ on S_j. Then C_j is a smooth closed subspace of X_j (cf. 10.3 (2)), and if $\sigma_{j+1}\colon M_{j+1} \to M_j$ is the blowing-up with centre C_j, then $X_{j+1,\mathrm{red}}$ is smooth (by 3.14) and $H_{X_{j+1},\cdot}$ is locally constant on $|X_{j+1}|$ (by 1.14 (1)). After finitely many blowings-up σ_i, $j < i \le k$, with such centres $C_i \subset S_i$, we get $S_k = \emptyset$; therefore, $X_{k,\mathrm{red}}$ is smooth, $H_{X_k,\cdot}$ is locally constant on $|X_k|$, and $X_{k,\mathrm{red}}$, E_k simultaneously have only normal crossings. $\square$

By Theorem 11.4, the Hilbert-Samuel space $S_{H_{X',a}}$ of X' (in Theorem 11.14) coincides with X'_{red} at every point a. If $\sigma\colon M' \to M$ denotes the composite of the blowings-up σ_j in Theorem 11.14, then $E' = \sigma^{-1}(\Sigma)$, where $\Sigma = \mathrm{Sing}_H X \cup \mathrm{Sing}\, X_{\mathrm{red}}$; in particular, σ restricts to an isomorphism over $\mathrm{Reg}\, X$. Of course, $X' = \emptyset$ if $\Sigma = |X|$.

Example 11.15. Let $X \subset \mathbb{R}^3$ denote the space (restriction to the $\mathbb{R}$-rational points of an affine scheme over $\mathbb{R}$) defined by $(y^2 - x^3)^2 + z^2 = 0$. Then X is a reduced hypersurface whose order (and therefore whose Hilbert-Samuel function) is constant. Clearly, $\Sigma = |X|$. But X can be desingularized in a meaningful sense by blowings-up over the origin.

Example 11.15 shows that the Hilbert-Samuel function itself is not a delicate enough invariant for resolution of singularities. But the desingularization algorithm determined by inv_X (in the proof of Theorem 11.14 above) nevertheless makes good sense because, for each j, we can ignore components C of the maximum locus of inv_X on Σ_j such that $C_a = |X|_a$ for all $a \in C$. The result will be: $H_{X'}$ locally constant on $|X'|$, $|X'| \neq \emptyset$ smooth, and $|X'|$, E' simultaneously normal crossings.

Consider a sequence of inv_X-admissible blowings-up (1.1). For each j, let $T_j = \mathrm{Sing}^{\mathrm{Zar}}_{\mathrm{inv}} X_j$ (defined in analogy with $\mathrm{Sing}^{\mathrm{Zar}}_H X_j$); then $|X_j|\backslash T_j$ is Zariski-open and dense in $|X_j|$. Let $a \in T_j$, for some j. If $|X_j|_a$ is irreducible, then (the germ at a of) $|S_{\mathrm{inv}_X(a)}| \subset T_{j,a}$ (as in the proof of Proposition 11.13). We do not know whether this is true in general. If it is, then there is a variant of the preceding result with each $C_j \subset T_j$. (In particular, $\sigma(E') \subset T_0$ has empty intersection with a Zariski-open and dense subset of $|X|$.)

12. How to avoid blowing up resolved points

Consider the algorithm used in Theorem 11.14. (It will be clear that the main result of this section, Theorem 12.4 below, applies to the other desingularization theorems of Sects. 10,11 as well.) Our invariant inv_X first prescribes a finite sequence of blowings-up $\sigma_{j+1}\colon M_{j+1} \to M_j$, $j = 0, \dots, k$, with smooth admissible centres C_j, such that:

(12.1) (1) For each j, $C_j \subset \Sigma_j = \mathrm{Sing}_H X_j \cup \mathrm{Sing}\, X_{j,\mathrm{red}}$.

(2) $\Sigma_{k+1} = \emptyset$; i.e., $X_{k+1,\mathrm{red}}$ is smooth and $H_{X_{k+1},\cdot}$ is locally constant on $|X_{k+1}|$.

Theorem 12.2. *Suppose that $X_{k+1} \neq \emptyset$. Then there is a further finite sequence of blowings-up $\sigma_{j+1}\colon M_{j+1} \to M_j$, $j = k+1, \dots, \ell$, with smooth centres C_j, such that:*

(1) For each j, $C_j \subset X_j \cap E_j$, C_j has only normal crossings with respect to E_j, and C_j includes no point where $X_{j,\mathrm{red}}$ and E_j simultaneously have only normal crossings.

(2) $X_{\ell+1,\mathrm{red}}$ and $E_{\ell+1}$ simultaneously have only normal crossings.

(It follows from (1) and (12.1) (2) that $X_{j+1,\mathrm{red}}$ is smooth and $H_{X_{j+1},\cdot}$ is locally constant on $|X_{j+1}|$, $j = k, \dots, \ell$.) Theorem 12.2 is a consequence of Theorem 12.4 below (applied with $M = M_{k+1}$, $X = X_{k+1,\mathrm{red}}$ and $E = E_{k+1}$).

Suppose $X \subset M$ are smooth spaces and that E is a collection of smooth hypersurfaces $H_i \subset M$, $i = -q, \dots, -1, 0$, such that E has only normal crossings. (We assume that, for all $a \in |X|$, all nonempty germs $H_{i,a}$ are distinct and none contains X_a.) We consider a sequence of transformations

$$
(12.3)\qquad
\begin{array}{ccccccccc}
\longrightarrow & M_{j+1} & \xrightarrow{\sigma_{j+1}} & M_j & \longrightarrow & \cdots & \longrightarrow & M_1 & \xrightarrow{\sigma_1} & M_0 = M \\
& X_{j+1} & & X_j & & & & X_1 & & X_0 = X \\
& E_{j+1} & & E_j & & & & E_1 & & E_0 = E
\end{array}
$$

such that, for each j: (1) σ_{j+1} is a (local) blowing-up with smooth centre $C_j \subset X_j \cap E_j$ such that C_j and E_j simultaneously have only normal crossings. (2) X_{j+1} is the strict transform of X_j by σ_j. (3) $E_{j+1} = \{H_{i,j+1} : i = -q, \dots, j+1\}$, where $H_{i0} = H_i$, $i = -q, \dots, 0$, $H_{i,j+1}$ is the strict transform of H_{ij} by σ_{j+1}, $i = -q, \dots, j$, and $H_{j+1,j+1} = \sigma_{j+1}^{-1}(C_j)$. (By (1), X_{j+1} is smooth and E_{j+1} has only normal crossings.)

Let $E_j^* = \{H_{ij} : i \le 0\}$, $j = 0, 1, \dots$. If $a \in M_j$, write $E(a) = \{H \in E_j : H \ni a\}$ and $E^*(a) = \{H \in E_j^* : H \ni a\}$. Each centre C_j will be chosen, more precisely, in the following way: For each $a \in C_j$, there will be a subset $F(a)$ of $E(a)$ containing $E(a)\backslash E^*(a)$, such that X_j, $F(a)$ simultaneously have normal crossings at a, C_j, $F(a)$ simultaneously have normal crossings at a, and $C_{j,a} \subset H_a$ for all $H \in E(a)\backslash F(a)$.

Theorem 12.4. *There is a sequence of blowings-up as in (12.3), such that:*

(1) For each j, $C_j \subset \Sigma_j^*$, where $\Sigma_j^* := \{a \in |X_j| \colon X_j$ and E_j do not simultaneously have normal crossings at $a\}$.

(2) $X_{\ell+1}$ and $E_{\ell+1}$ simultaneously have only normal crossings.

Lemma 12.5. *For each j, Σ_j^* is a (nowhere dense) Zariski-closed subset of $|X_j|$.*

Proof. For all $\Lambda \subset E_j$, set $X_j(\Lambda) = X_j \cap \bigcap_{H\in\Lambda} H$. Then $\Sigma_j^* = \bigcup_{\Lambda\subset E_j}\{a \in |X_j(\Lambda)| \colon e_{X_j(\Lambda),a} > d(a) - \#\Lambda$, where $d(a) = \dim X_{j,a}\}$. Since $d(a)$ is locally constant on $|X_j|$, Σ_j^* is Zariski-closed, by Remarks 9.1. □

Remarks 12.6. For each j and $i = -q, \ldots, j$, $X_j \cap H_{ij}$ is a hypersurface in X_j; let $\mathscr{I}_{ij} \subset \mathscr{O}_{X_j}$ denote the corresponding (principal) ideal. Let $a \in |X_j|$. Then X_j and H_{ij} have normal crossings at a if and only if $\nu_{\mathscr{I}_{ij},a} = 0$ or 1 (cf. Remark 1.8). Clearly, $a \notin \Sigma_j^*$ if and only if: (1) $\nu_{\mathscr{I}_{ij},a} = 0$ or 1, $i = -q, \ldots, 0$. (This condition is automatically satisfied for $i = 1, \ldots, j$.) (2) $X_j \cap E_j := \{X_j \cap H_{ij} : i = -q, \ldots, j\}$ has normal crossings at a, and any two nonempty germs $(X_j \cap H_{ij})_a$ are distinct.

Note that if $C_j \subset X_j \cap H_{ij} = V(\mathscr{I}_{ij})$, then $\mathscr{I}_{i,j+1}$ is the transform $y_{\mathrm{exc}}^{-1}\sigma_{j+1}^{-1}(\mathscr{I}_{ij}) = [\sigma_{j+1}^{-1}(\mathscr{I}_{ij}) : y_{\mathrm{exc}}]$ of $\mathscr{I}_{ij}$ by (the restriction to X_{j+1} of) the blowing-up σ_{j+1} (cf. Proposition 3.13 ff.); i.e., $\mathscr{I}_{i,j+1,b}$, $b \in \sigma_{j+1}^{-1}(a)$, is given by the transform of the infinitesimal presentation $\{(f, 1)\}$, where f denotes a generator of $\mathscr{I}_{ij,a}$.

Proof of Theorem 12.4. We argue by induction on $\dim X$. Consider any sequence of transformations (12.3). Write $\mathscr{J}_{i0} = \mathscr{I}_{i0}$, $i = -q, \ldots, 0$. For each j and $i = -q, \ldots, j$, let $\mathscr{J}_{i,j+1}$ denote the strict transform of $\mathscr{J}_{ij}$ by σ_{j+1} (more precisely, by $\sigma_{j+1}|X_{j+1}$); i.e., $\mathscr{J}_{i,j+1} = \sum_{k\geq 0}[\sigma_{j+1}^{-1}(\mathscr{J}_{ij}) : y_{\mathrm{exc}}^k]$. Then $\mathscr{I}_{ij} = \mathscr{J}_{ij}$, $i = 1, \ldots, j$, and $\mathscr{I}_{ij} = \mathscr{D}_{ij} \cdot \mathscr{J}_{ij}$, $i = -q, \ldots, 0$, where $\mathscr{D}_{ij}$ is a product of exceptional divisors $\mathscr{I}_{hj}$, $h = 1, \ldots, j$. If $a \in |X_j|$ and f_i, g_i denote generators of $\mathscr{I}_{ij,a}$, $\mathscr{J}_{ij,a}$ (respectively), $i = -q, \ldots, 0$, then $f_i = D_i \cdot g_i$, where D_i is a monomial in generators of $\mathscr{I}_{hj,a}$, $h = 1, \ldots, j$. (D_i is the greatest factor of f_i having this form.)

Set $\mathscr{J}_j = \mathscr{J}_{-q,j} \cdot \cdots \cdot \mathscr{J}_{0j}$, $j = 0, 1, \ldots$. Let $a \in |X_0|$. Write $\nu_1(a) = \nu_{\mathscr{J}_0,a}$. If $g = g_{-q} \cdot \cdots \cdot g_0$, where g_i generates $\mathscr{J}_{i0}$, $i = -q, \ldots, 0$, then $\nu_1(a) = \mu_a(g)$ and $\mathscr{G}_a = \{(g, \mu_a(g))\}$ is a presentation of ν_1 at a of codimension 0 (in X_0). We can use the construction of Chapter II to extend $\mathrm{inv}_{1/2} = \nu_1$ to an invariant $\mathrm{inv}_{\mathscr{J}}$ which is defined inductively over a sequence of (local) blowings-up $\sigma_{j+1}\colon X_{j+1} \to X_j$ and successive transforms $\mathscr{J}_{i,j+1}$ of $\mathscr{J}_{ij}$, $i = -q, \ldots, 0$, where the successive centres of blowing up are $\mathrm{inv}_{\mathscr{J}}$-admissible. If $a \in |X_j|$, then $\mathrm{inv}_{1/2}(a) = \nu_{\mathscr{J}_j,a}$.

Consider any such sequence. Let $a \in |X_j|$ and let $t(a)$ denote the number of distinct ideals $\mathscr{J}_{ij,a} \subset \mathscr{O}_{X_j,a}$, $i = -q, \ldots, 0$, such that $\nu_{\mathscr{J}_{ij},a} \geq 1$. Clearly, $\nu_{\mathscr{J}_j,a} \geq t(a)$, and $\nu_{\mathscr{J}_j,a} = t(a)$ if and only if all proper ideals $\mathscr{J}_{ij,a}$, $i = -q, \ldots, 0$, are distinct and of order 1. Set $Z_j := \{x \in |X_j| : \nu_{\mathscr{J}_j,x} > t(x)\}$. Then Z_j is a Zariski-closed subset of $|X_j|$. If $a \in Z_j$, then the germ $S_{\mathrm{inv}_{\mathscr{J}}}(a) \subset Z_j$ (since already $S_{\mathrm{inv}_{1/2}}(a) \subset Z_j$).

We can obtain such a sequence of $\mathrm{inv}_{\mathscr{J}}$-admissible transformations of X and the $\mathscr{J}_{i0}$ by choosing as each successive centre C_j the locus of the (finitely many) maximal values of the (extended) invariant $\mathrm{inv}^e_{\mathscr{J}}$ on Z_j. The blowings-up σ_{j+1}: $M_{j+1} \to M_j$ with centres C_j form a sequence (12.3) where each $C_j \subset \Sigma^*_j$ (and if $a \in C_j$, then $C_{j,a} \subset H_{ij,a}$ for all $H_{ij} \in E^*(a)$). After finitely many blowings-up, we get $Z_k = \emptyset$. Then all $V(\mathscr{J}_{ik})$ are smooth and, for each $a \in |X_k|$, all proper ideals $\mathscr{J}_{ik,a}$ are distinct.

Now, by induction on $\dim X$, we can assume that Theorem 12.4 holds replacing M, X and E by M_k, $V(\mathscr{J}_{0k}) \subset X$ and $E_k \backslash \{H_{0k}\}$. Then the corresponding sequence of transformations of M_k, X_k and E_k also satisfies (12.3) and 12.4(1). (The essential point is this: Let $a \in V(\mathscr{J}_{0k})$. Since all proper ideals $\mathscr{J}_{ik,a}$ are distinct, it follows that $V(\mathscr{J}_{0k}) \not\subset H_{ik}$, $i \neq 0$, and (by Lemma 12.7 below) that if F is a subset of $E_k \backslash \{H_{0k}\}$, and $V(\mathscr{J}_{0k})$ and F simultaneously have normal crossings at a, then so do X_k and F.)

Lemma 12.7. *Let $W \subset Y \subset N$ be smooth spaces, and let F be a collection of smooth hypersurfaces in N. Let $a \in |N|$. If $W_a \not\subset H_a$, for all $H \in F$, and W and F simultaneously have normal crossings at a, then so do Y and F.*

The proof of the lemma is simple. The result of our inductive application of Theorem 12.4 above is that we can assume, in addition to the preceding conditions, that the $V(\mathscr{J}_{ik})$, $i = -q, \ldots, k$, simultaneously have normal crossings at each point of $V(\mathscr{J}_{0k})$. We can then apply Theorem 12.4 (by induction) to M_k, $V(\mathscr{J}_{-1,k})$ and $E_k \backslash \{H_{-1,k}\}$, and afterwards continue for each $i = -2, \ldots, -q$, to arrive at the assumptions of the combinatorial Lemma 12.8 following. (In fact, in addition to the hypotheses of Lemma 12.8, we have: For each $i = -q, \ldots, 0$ and each $a \in V(\mathscr{J}_{ik})$, $\mathscr{I}_{hk,a} = \mathscr{J}_{hk,a}$, $i \neq h \leq 0$, and $\mathscr{J}_{hk,a} = \mathscr{O}_{X_k,a}$ if $\mathscr{D}_{ik,a}$ is a proper ideal. We will not use this extra information.)

The combinatorial invariant in the proof of 12.8 is a refinement of that in Theorem 1.13, for the purpose of the stronger conditions required on the centres of blowing up.

Lemma 12.8. *Suppose that, for each $a \in |X_k|$: (1) $\nu_{\mathscr{J}_{ik,a}} = 0$ or 1, for all i. (2) The $\mathscr{J}_{ik}$, $i = -q, \ldots, k$, simultaneously have normal crossings at a, and all proper ideals $\mathscr{J}_{ik,a}$ are distinct. Then there is a sequence of blowings-up σ_{j+1}, $j = k, \ldots, \ell$, as in (12.3) with centres C_j* admissible *in the stronger sense that each C_j is the intersection of the $V(\mathscr{J}_{ij})$ for certain $i > 0$, such that the conclusion of Theorem 12.4 holds (i.e., $C_j \subset \Sigma^*_j$, $j = k, \ldots, \ell$, and $\Sigma^*_{\ell+1} = \emptyset$).*

Proof. Consider any sequence of blowings-up σ_{j+1}, $j = k, \ldots$, whose centres are admissible in the sense of the lemma. For each j, the transforms $\mathscr{I}_{i,j+1}$ of the $\mathscr{I}_{ij} = \mathscr{D}_{ij}\mathscr{J}_{ij}$, $i \leq 0$, are given by $\mathscr{I}_{i,j+1} := y_{\mathrm{exc}}^{-1}\sigma_{j+1}^{-1}(\mathscr{I}_{ij}) = \mathscr{D}_{i,j+1} \cdot \mathscr{J}_{i,j+1}$, where $\mathscr{D}_{i,j+1} = y_{\mathrm{exc}}^{-1}\sigma_{j+1}^{-1}(\mathscr{D}_{ij})$ (because $C_j \not\subset V(\mathscr{J}_{ij})$, $i \leq 0$).

If $a \in |X_j|$, we define $\tau_{\mathscr{D}}(a) := \# I(a)$, where $I(a) := \{i = -q, \ldots, 0 : \mathscr{I}_{ij,a} \neq \mathscr{J}_{ij,a}\}$. Then $\Sigma_j^* = \{x \in |X_j| : \tau_{\mathscr{D}}(a) > 0\}$. Clearly, $\tau_{\mathscr{D}}$ is Zariski-semicontinuous on $|X_j|$. Let $a \in |X_j|$. Then $\tau_{\mathscr{D}}$ admits a presentation at a (of codimension 0 in X_j) given by $\mathscr{H}_0(a) := \{(D_i, 1) : i \in I(a)\}$, where D_i is a generator of $\mathscr{D}_{ij,a}$. (We use "presentation" here in the weaker sense of "with respect to transformations of type (i)" only (cf. Definition 4.6), where the centres of blowing up are admissible in the stronger sense above. Invariance will be automatic from the combinatorial definitions, so that transformations of types (ii), (iii) are unneeded.) Let D_{*1} denote the greatest common divisor of the D_i, $i \in I(a)$. Write $D_i = D_{*1} \cdot f_i$, $i \in I(a)$. We introduce the invariant $\nu_1(a) := \min \mu_a(f_i)$ and define $\mathrm{inv}_1(a) := \big(\tau_{\mathscr{D}}(a), \nu_1(a)\big)$. Obviously, $0 \leq \nu_1(a) < \infty$. If $\nu_1(a) = 0$, we set $\mathrm{inv}_{\mathscr{D}}(a) := \mathrm{inv}_1(a)$; in this case, $\mathscr{F}_1(a) := \{(D_{*1}, 1)\}$ is a codimension 0 presentation of $\mathrm{inv}_{\mathscr{D}}$ at a. Assume $\nu_1(a) > 0$ (i.e., $\nu_1(a) \geq 1$, since $\nu_1(a) \in \mathbb{N}$). Then $\mathscr{F}_1(a) := \{(f, \mu_f) = \big(f_i, \nu_1(a)\big) : i \in I(a)\}$ is a codimension 0 presentation of inv_1 at a, and $\mu_{\mathscr{F}_1(a)} = 1$.

We can extend inv_1 to an invariant $\mathrm{inv}_{\mathscr{D}}$ essentially using the construction of Chapter II, with $E^r(a) = \emptyset$ for all r (so there are no terms s_r in $\mathrm{inv}_{\mathscr{D}}$): Let $a \in |X_j|$ and let x_i denote a local generator at a of the ideal $\mathscr{I}_{ij,a} = \mathscr{J}_{ij,a}$ of $X_j \cap H_{ij}$, $i > 0$. For each $(f, \mu_f) \in \mathscr{F}_1(a)$, we can write $f = \prod_{i>0} x_i^{\alpha_i(f)}$ (up to an invertible factor). Since $\mu_{\mathscr{F}_1(a)} = 1$, $\sum_{i>0} \alpha_i(f) = \mu_f$ for some f, so that $\mathscr{F}_1(a) \sim \mathscr{F}_1(a) \cup \{(x_i, 1)\}$ for some $i > 0$. Let i_0 denote the least such i. Then inv_1 admits an equivalent codimension 1 presentation $\big(N_1(a), \mathscr{H}_1(a)\big)$, where $N_1(a) = H_{i_0 j,a}$ and $\mathscr{H}_1(a) = \Big\{\Big(\prod_{i \neq i_0} x_i^{\alpha_i(f)}, \mu_f - \alpha_{i_0}(f)\Big) : (f, \mu_f) \in \mathscr{F}_1(a)\Big\}$. We define $\nu_2(a) := \mu_2(a) - \sum_{i_0 \neq i > 0} \mu_{2,H_{ij}}(a)$, where $\mu_2(a) := \mu_{\mathscr{H}_1(a)}$ and each $\mu_{2,H_{ij}}(a) := \min\{\mu_{H_{ij},a}(h)/\mu_h : (h, \mu_h) \in \mathscr{H}_1(a)\}$. Thus $0 \leq \nu_2(a) < \infty$. We set $\mathrm{inv}_2(a) := \big(\mathrm{inv}_1(a); \nu_2(a)\big)$; inv_2 admits a codimension 1 presentation $\big(N_1(a), \mathscr{F}_2(a)\big)$ at a, where $\mathscr{F}_2(a) := \Big\{\big(D_{*2}^{-\mu_h} \cdot h,\ \mu_h \cdot \nu_2(a)\big)$, for all $(h, \mu_h) \in \mathscr{H}_1(a)$, together with $\big(D_{*2}, 1 - \nu_2(a)\big)\Big\}$ and $D_{*2} := \prod_{i \neq i_0} x_i^{\mu_{2,H_{ij}}(a)}$ (cf. Ch. II). In the case that $\nu_2(a) > 0$, it follows that $\mu_{\mathscr{F}_2(a)} = 1$. The construction can be repeated in increasing codimension until eventually $\nu_{t+1}(a) = 0$; then $\mathrm{inv}_{\mathscr{D}}(a) := \mathrm{inv}_{t+1}(a)$.

We can obtain a sequence of blowings-up σ_{j+1}, $j = k, \ldots$, satisfying the conditions of the lemma by choosing as each successive centre C_j the locus of the (finitely many) maximal values of the (extended) invariant $\mathrm{inv}_{\mathscr{D}}^e$ on Σ_j^*. For some j, say $j = \ell$, we get $\Sigma_{\ell+1}^* = \emptyset$. We have thus proved the lemma and Theorem 12.4. $\square$

Example 12.9. Let $X = V(x_3 - x_1 x_2) \subset \underline{k}^3 = M$, and let $E = \{H_0\}$, where $H_0 = V(x_3)$ (cf. Example 2.3, Year one). Clearly, $\Sigma_0^* = \{0\}$. If we use (x_1, x_2) as coordinates on X, then $X \cap H_0 = V(\mathscr{I})$, where $\mathscr{I} \subset \mathscr{O}_X$ is the ideal generated by $x_1 x_2$. Theorem 12.4 prescribes $C_0 = \{0\}$ as the centre of the first blowing-up

$\sigma_1\colon M_1 \to M_0 = M$. Then $E_1 = \{H_{01}, H_{11}\}$, where $H_{01} = H_0'$ and $H_{11} = \sigma_1^{-1}(C_0)$, so that $\Sigma_1^* = X_1 \cap H_{01} \cap H_{11}$. The hypotheses of the combinatorial Lemma 12.8 are satisfied by X_1, E_1; the lemma prescribes $C_1 = \Sigma_1^*$ as the centre of the next blowing-up $\sigma_2\colon M_2 \to M_1$. Then $\Sigma_2^* = \emptyset$; i.e., X_2 and E_2 simultaneously have only normal crossings.

13. Universal desingularization; canonical desingularization of non-compact analytic spaces

Let $\mathscr{A}$ denote a class of spaces in (0.2) (1) or (2). Let $X = (|X|, \mathscr{O}_X) \in \mathscr{A}$. Then X can be locally embedded in a manifold; i.e., each $a \in |X|$ admits an open neighbourhood U in X such that $X|U$ can be embedded as a closed subspace of a smooth space $M \in \mathscr{A}$. If $\sigma\colon X' \to X$ is a blowing-up of X with (smooth) centre C, then, for any local embedding $X|U \hookrightarrow M$, $\sigma\colon X'|\sigma^{-1}(U) \to X|U$ can be identified with the morphism $(X|U)' \to X|U$ induced by the blowing-up $\pi\colon M' \to M$ with centre $C \cap U$, where $(X|U)'$ is the strict transform of $X|U$ by π. Moreover, $\sigma\colon X' \to X$ is uniquely determined (up to equivalence) by this condition.

Our invariant inv_X (as defined in Chapters II, III) *a priori* depends on a pair (X, M), where M is a manifold and X is a closed subspace of M. However:

Remarks 13.1. (1) Consider two such pairs (X^i, M^i), $i = 1, 2$, and an isomorphism $\varphi\colon M^1 \to M^2$ such that $\varphi(X^1) = X^2$. Write $(X_0^i, M_0^i) = (X^i, M^i)$ and $E_0^i = \emptyset$, $i = 1, 2$, and set $\varphi_0 = \varphi$. Suppose we have a sequence of inv_{X^1}-admissible transformations of M_0^1, X_0^1, E_0^1 as in (1.1). (We will write $(M_{j+1}^1; X_{j+1}^1, E_{j+1}^1) \xrightarrow{\sigma_{j+1}^1} (M_j^1; X_j^1, E_j^1)$, $j = 0, 1, \ldots,$ to save space.) Then (by invariance of inv), there is a sequence of inv_{X^2}-admissible transformations $(M_{j+1}^2; X_{j+1}^2, E_{j+1}^2) \xrightarrow{\sigma_{j+1}^2} (M_j^2; X_j^2, E_j^2)$ and, for each j, an isomorphism $\varphi_j\colon M_j^1 \to M_j^2$ such that $\varphi_j \circ \sigma_{j+1}^1 = \sigma_{j+1}^2 \circ \varphi_{j+1}$, $\varphi_j(X_j^1) = X_j^2$, $\varphi_j(E_j^1) = E_j^2$ and, moreover, $\mathrm{inv}_{X^1}(a) = \mathrm{inv}_{X^2}\left(\varphi_j(a)\right)$ for all $a \in X_j^1$ (cf. Remarks 9.15).

(2) Consider $X \hookrightarrow M$ and an embedding $\iota\colon M \hookrightarrow N$ of M (as a closed submanifold of a manifold N). Write $(X_0, M_0) = (X, M)$, $(Y_0, N_0) = \left(\iota(X), N\right)$, $E_0 = F_0 = \emptyset$, and $\iota_0 = \iota$. Suppose we have a sequence of inv_X-admissible transformations $(M_{j+1}; X_{j+1}, E_{j+1}) \xrightarrow{\sigma_{j+1}} (M_j; X_j, E_j)$ (where each $\sigma_{j+1}\colon M_{j+1} \to M_j$ is a blowing-up of M_j with smooth centre C_j) as in (1.1). It follows from our constructive definition of inv_X that there is a sequence of inv_Y-admissible transformations $(N_{j+1}; Y_{j+1}, F_{j+1}) \xrightarrow{\tau_{j+1}} (N_j; Y_j, F_j)$ and, for each j, an embedding $\iota_j\colon M_j \hookrightarrow N_j$, such that for each j, $\tau_{j+1}\colon N_{j+1} \to N_j$ is the blowing-up of N_j with centre $D_j = \iota_j(C_j)$, $\iota_j \circ \sigma_{j+1} = \tau_{j+1} \circ \iota_{j+1}$, $Y_j = \iota(X_j)$, M_j and F_j simultaneously have only normal crossings, $\iota_j(E_j) = \{M_j \cap H : H \in F_j\}$ and, moreover, $\mathrm{inv}_X(a) = \mathrm{inv}_Y\left(\iota_j(a)\right)$ for all $a \in |X_j|$ (cf. Remarks 9.15).

Universal embedded resolution of singularities. Given $X \hookrightarrow M$ in $\mathscr{A}$ and an inv_X-admissible sequence (1.1), let inv_X^e be the extended invariant as in Remark 1.16.

Consider $X \in \mathscr{A}$ (not necessarily globally embedded). If $a \in |X|$, then there is a local embedding $X|U \hookrightarrow M$ at a in a manifold M of dimension $e_{X,a} = H_{X,a}(1) - 1$; $e_{X,a}$ is the minimal embedding dimension and any two such minimal embeddings are locally related by an isomorphism as in Remarks 13.1 (1).If $a \in |X|$, then $\mathrm{inv}_X(a)$ can be defined using any local embedding $X|U \hookrightarrow M$ over a neighbourhood U of a; by 13.1, $\mathrm{inv}_X(a)$ is independent of the choice of local embedding. (In fact, $\mathrm{inv}_X(a)$ depends only on $\widehat{\mathscr{O}}_{X,a}$.) Set $X_0 = X$ and let $\sigma_1\colon X_1 \to X_0$ denote a blowing-up with centre a smooth inv_X-admissible subspace C_0 of $X_0 = X$. By 13.1, inv_X is defined on X_1, independently of a choice of local embedding of X, and so on. In other words, the definition of inv_X over an admissible sequence of blowings-up $\sigma_{j+1}\colon X_{j+1} \to X_j$ extends to this context. Clearly, the analogue of Theorem 1.14 is true (where property (3) can be understood in terms of any local embedding, or formally in terms of a surjective homomorphism to $\widehat{\mathscr{O}}_{X,a}$ from a complete regular local ring).

Suppose that $|X|$ is quasi-compact (or, in the case of analytic spaces, that X is the restriction of an analytic space to a relatively compact open subset). Then the desingularization algorithm of Theorem 1.6 (or 10.7 or 11.14, as the case may be) applies to X: The compactness hypothesis guarantees that inv_X takes only finitely many values on each successive transform of X. For any local embedding $X|U \hookrightarrow M$, the sequence of blowings-up of $X|U$ induced by that of X is that which is given by the proof of 1.6 applied to $X|U \hookrightarrow M$. By the remarks above, over each $a \in |X|$, inv_X and the desingularization algorithm depend only on $\widehat{\mathscr{O}}_{X,a}$. We obtain the following theorem:

Theorem 13.2. *(1) There is a finite sequence of blowings-up $\sigma_{j+1}\colon X_{j+1} \to X_j$, where $X_0 = X$, such that, for any local embedding $X|U \hookrightarrow M$ of X, the sequence of blowings-up σ_{j+1} restricted to the inverse images of U is induced by embedded desingularization of $X|U$ in the sense of Theorem 1.6 (or Theorem 10.7 or 11.14, as the case may be).*

(2) The desingularization is **universal** *in the sense that, to each X in $\mathscr{A}$ satisfying the compactness hypothesis above, we associate a morphism $\sigma_X\colon X' \to X$ such that:*

(i) σ_X is a composite of a finite sequence of blowings-up as in (1).

(ii) Let X and Y be two spaces satisfying the compactness hypothesis, and let $\varphi\colon X|U \xrightarrow{\cong} Y|V$ be an isomorphism over open subsets U, V of $|X|$, $|Y|$ (respectively). Then there is an isomorphism $\varphi'\colon X'|\sigma_X^{-1}(U) \xrightarrow{\cong} Y'|\sigma_Y^{-1}(V)$ such that the diagram

$$
\begin{array}{ccc}
X'|\sigma_X^{-1}(U) & \xrightarrow{\varphi'} & Y'|\sigma_Y^{-1}(V) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & Y
\end{array}
$$

commutes. (The lifting φ' of φ is necessarily unique.) In fact, φ lifts to isomorphisms throughout the entire desingularization towers.

Theorem 12.2 on avoiding blowing-up resolved points can be incorporated in 13.2.

Canonical desingularization of analytic spaces. Finally, we consider desingularization of an analytic space $X = (|X|, \mathcal{O}_X)$ defined over a locally compact field $\underline{k}$ of characteristic zero (i.e., $\underline{k} = \mathbb{R}, \mathbb{C}$, or a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, where p is prime. In the latter case, X is locally a subspace of a manifold in the sense of Serre [Se].) We assume that $|X|$ is countable at infinity. In general, of course, X cannot be desingularized by a finite sequence of global blowings-up with smooth centres. There are two natural ways to generalize Theorem 13.2 above:

Theorem 13.3. *There is a morphism $\sigma_X\colon X' \to X$ such that, for any relatively compact open subset U of $|X|$, the restriction $\sigma_{X|U}\colon X'|\sigma_X^{-1}(U) \to X|U$ is a composite of a finite sequence of blowings-up as in Theorem 13.2 (1) (where the latter is formulated using either Theorem 1.6 or 11.14 locally). Desingularization of analytic spaces in this sense is universal (as in 13.2 (2)).*

Proof. This follows from Theorem 13.2: To every relatively compact open subset U of $|X|$, we associate a morphism $\sigma_{X|U}\colon (X|U)' \to (X|U)$, where $\sigma_{X|U}$ is the composite of a finite sequence of blowings-up as in 13.2(1), so that the universality condition (2) is satisfied. Thus, if $U \subset V$ are relative compact open subsets of $|X|$, the inclusion $X|U \hookrightarrow X|V$ lifts to $(X|U)' \hookrightarrow (X|V)'$, and σ_X is given by the direct limit. □

The morphism σ_X of Theorem 13.3 is not defined as a composite of global blowings-up with smooth centres. We can obtain resolution of singularities of analytic spaces in this stronger way, but at the expense of weakening the notion of universality:

We say that a sequence of blowings-up $\longrightarrow X_{j+1} \xrightarrow{\sigma_{j+1}} X_j \longrightarrow \cdots \longrightarrow X_0 = X$ is *locally finite* if all but finitely many of the blowings-up σ_{j+1} are trivial over any compact subset of $|X|$. The composite of a locally finite sequence of blowings-up is a well-defined morphism $\sigma\colon X' \to X$.

Theorem 13.4. *There is a locally finite sequence of blowings-up $\sigma_{j+1}\colon X_{j+1} \to X_j$ with smooth* inv_X*-admissible centres $C_j \subset X_j$ (where $X_0 = X$) such that*

(1) For each j, either $C_j \subset \operatorname{Sing} X_j$ or X_j is smooth and $C_j \subset E_j$.

(2) Let $\sigma\colon X' \to X$ denote the composite of the sequence of blowings-up σ_{j+1}. Then X' is smooth, and X', E' simultaneously have only normal crossings (where E' denotes the collection of all exceptional divisors).

(3) σ is **canonical** *in the sense that any isomorphism $\varphi\colon X|U \xrightarrow{\cong} X|V$, where U and V are open subsets of $|X|$, lifts to an isomorphism $\varphi'\colon X'|\sigma^{-1}(U) \to X'|\sigma^{-1}(V)$.*

The assertions concerning E_j and E' can be understood locally, for example as in Theorem 1.6, using an embedding $X|U \hookrightarrow M$ over a relatively compact open subset of $|X|$. Of course, X' may be empty (as in 1.6); Theorem 13.4 is a meaningful geometric desingularization theorem at least in the case that X is geometric (Definition 10.5). Every reduced complex analytic space is geometric. Our proof below can also be used to extend Theorem 1.10 to the non-compact analytic case, in general.

Proof of Theorem 13.4. We will use the algorithm of Theorem 1.6 locally (over a relatively compact open subset U of $|X|$, say), extending each centre of blowing-up to a global analytic subspace of X and desingularizing this subspace (using induction on dimension) by a locally finite sequence of inv_X-admissible blowings-up which are trivial over U. As in the proof of 1.6 (Sect. 10), we first find a locally finite sequence of blowings-up σ_{j+1} with inv_X-admissible centres $C_j \subset \mathrm{Sing}\, X_j$ such that, if $\sigma\colon X' \to X$ denotes the composite of the sequence, then X' is smooth. Afterwards, we repeat the algorithm using $\{x : \ s_1(x) > 0\}$ instead of $\mathrm{Sing}\, X_j$ at every stage (as in Sect. 10) to achieve the normal crossings condition. We will describe only the first of the two steps.

Since $|X|$ can be exhausted by a sequence of relatively compact open subsets, it is enough to prove the following assertion: Let U be a relatively compact open subset of $|X|$. Then there is a locally finite sequence of blowings-up σ_{j+1}: $X_{j+1} \to X_j$ with smooth inv_X-admissible centres $C_j \subset X_j$, satisfying (3) of the theorem as well as:

(1′) Each $C_j \subset \mathrm{Sing}\, X_j$.

(2′) If $\sigma\colon X' \to X$ denotes the composite of the σ_{j+1}, then $\mathrm{Sing}(X'|\sigma^{-1}(U)) = \emptyset$.

To prove this assertion, consider the resolution algorithm of Theorem 1.6 applied to $X|U$ (as in Theorem 13.2); say that

$$\text{(13.5)} \qquad \cdots \longrightarrow (X|U)_{j+1} \longrightarrow (X|U)_j \longrightarrow \cdots \longrightarrow (X|U)_0 \;=\; X|U$$

is the sequence of blowings-up given by the algorithm (for the first of the two steps; i.e., to reduce to the case that $\mathrm{Sing}(X|U)' = \emptyset$). We can assume that each centre of blowing up C_j is pure-dimensional (by using, for example, as each successive C_j, the lowest-dimensional components of the maximum locus of $\mathrm{inv}^e_{X|U}$ in $\mathrm{Sing}(X|U)_j$, where the extended invariant inv^e_X is fixed as in Remark 1.16). Suppose we have a (locally finite) sequence of inv_X-admissible blowings-up of X satisfying (1′) and (3) and restricting, over U, to part of the resolution tower (13.5); let us say, to that part up to $(X|U)_j$ (apart from blowings-up that are trivial over U). Let $\sigma^0\colon X^0 \to X$ denote the composite of this sequence of blowings-up of X, so that the restriction $\sigma^0\colon X^0|(\sigma^0)^{-1}(U) \to X|U$ can be identified with the composite $(X|U)_j \to X|U$ in (13.5).

Set $V = (\sigma^0)^{-1}(U)$. Let Λ denote the (finite) set of maximal values $\lambda = \mathrm{inv}^e_X(a)$ of inv^e_X on $\mathrm{Sing}(X^0|V)$, and let $S := \bigcup_{\lambda\in\Lambda}\{x \in \mathrm{Sing}(X^0|V) :\ \mathrm{inv}^e_X(a) = \lambda\}$. Then S defines a smooth closed subspace of $X^0|V$ (by Remarks 10.3, as in the proof of Theorem 1.6). For each $\lambda \in \Lambda$, let T_λ denote the smallest closed analytic subspace of X^0 such that $|T_\lambda| = \{x \in \operatorname{Sing} X^0 : \operatorname{inv}_X^e(x) \geq \lambda\}$. (The latter is Zariski-closed since inv_X^e is Zariski-semicontinuous; cf. Remark 6.14.) Let S_q denote the union of the components of S of the smallest dimension q. Let T denote the union of the components of $\bigcup_{\lambda\in\Lambda} T_\lambda$ of dimension q; T is a well-defined closed analytic subspace of X^0. Clearly, $T|V = S_q$ and $\bigcup_{\lambda\in\Lambda}\{x \in |T| : \operatorname{inv}_X^e(x) = \lambda\}$ is Zariski-open in $|T|$.

Our aim is to desingularize $X^1 = T$ by a locally finite sequence of inv_X-admissible blowings-up of X^0 which is trivial over V and satisfies (1′) and (3), such that inv_X is locally constant on the final (smooth) transform T' of T. Then T' provides a centre for a blowing-up that restricts to $(X|U)_{j+1} \to (X|U)_j$ in (13.5), and the theorem follows recursively. We achieve the aim by an inductive construction, for the purpose of which we formulate a more general problem. (The result needed is Lemma 13.7 with $k = 1$).

Consider a decreasing chain of pure-dimensional closed analytic subspaces of X^0, $X^1 \supset X^2 \supset \cdots \supset X^k$, $k \geq 1$, such that $|X^1| \subset \operatorname{Sing} X^0$, $\dim X^1 < \infty$, and $\dim X^{i+1} < \dim X^i$, $i = 1, \ldots, k-1$. Assume that each X^i is preserved by the liftings of local isomorphisms of X given by the canonicity condition on σ^0: $X^0 \to X$. Consider a locally finite sequence of inv_X-admissible blowings-up

$$\text{(13.6)} \qquad \cdots \longrightarrow X_{j+1}^0 \xrightarrow{\sigma_{j+1}} X_j^0 \longrightarrow \cdots \longrightarrow X_0^0 = X^0$$

with smooth centres $C_j \subset X_j^0$, where for each $i = 1, \ldots, k$ and each $j = 0, 1, \ldots$, X_{j+1}^i denotes the smallest closed analytic subspace of X_{j+1}^0 containing $\sigma_{j+1}^{-1}(X_j^i)\backslash\sigma_{j+1}^{-1}(C_j)$. Then, for each j, $\operatorname{Sing} X_j^0 \supset |X_j^1|$, $X_j^1 \supset \cdots \supset X_j^k$, and X_j^i is pure-dimensional and $\dim X_j^{i+1} < \dim X_j^i$, $i \geq 1$. For each j, we define $\iota^k(a) := \left(\operatorname{inv}_X(a), H_{X_j^1,a}, \ldots, H_{X_j^k,a}\right)$, $a \in |X_j^0|$. (Say that $\iota^k(a) \leq \iota^k(b)$ means componentwise $\leq$.)

Lemma 13.7. *There is a locally finite sequence of blowings-up (13.6) with smooth ι^k-admissible centres $C_j \subset X_j^k$ such that:*
(1) For each j, C_j is pure-dimensional and C_j includes no point a at which X_j^k is smooth and ι_k is locally constant on $|X_j^k|$.
(2) For each $i = 0, \ldots, k$, let $(X^i)'$ denote $\varprojlim X_j^i$. (In particular, the induced morphism σ: $(X^0)' \to X^0$ is the composite of the σ_{j+1}.) Then $(X^k)'$ is smooth and $\left(\operatorname{inv}_X, H_{(X^1)',\cdot}, \ldots, H_{(X^k)',\cdot}\right)$ is locally constant on $|(X^k)'|$.
(3) $\sigma^0 \circ \sigma$: $(X^0)' \to X$ satisfies the canonicity condition 13.4 (3), and each $(X^i)'$ is preserved by the liftings of the local isomorphisms of X given by canonicity.

Proof. Our proof is by induction on $\dim X^k$. (The case of dimension zero is trivial.) It is enough to prove the following assertion: Let U be a relatively compact open subset of $|X|$, and let $V = (\sigma^0)^{-1}(U)$. Then there is a locally finite sequence of blowings-up (13.6) with smooth ι^k-admissible centres $C_j \subset X_j^k$ such that (1) and (3) of the lemma hold, and (2) holds on the inverse image of V.

Write $\underline{X} := (X^0, X^1, \ldots, X^k)$. Given a sequence (13.6), we define $\operatorname{inv}_{1/2}(a) := \iota^k(a)$, $a \in |X_j^0|$. (In the analytic case here) it is easy to see that each term of

$\iota^k(\cdot)$ admits a (semicoherent) codimension zero presentation; the union of these presentations is a codimension zero presentation of $\iota^k(\cdot)$. Therefore, we can use the construction of Sect. 6 to extend $\mathrm{inv}_{1/2}$ to an invariant $\mathrm{inv}_{\underline{X}}(a)$, provided that the centres of blowing up C_j are chosen successively to be $\mathrm{inv}_{\underline{X}}(a)$-admissible. Since $\iota^k(\cdot)$ includes $H_{X_j^k,\cdot}$, $\mathrm{inv}_{\underline{X}}$ is also locally constant on $|X_j^k|$ at a point a as in (1).

We can use an analogue for $\mathrm{inv}_{\underline{X}}$ of the algorithm of Theorem 1.6 to prove Lemma 13.7 for $\underline{X}|V$: For each j, let V_j denote the inverse image of V in $|X_j^0|$. We can take as each successive centre the smallest dimensional components of maximum locus of $\mathrm{inv}^e_{\underline{X}}$ on Y_j^k, where Y_j^k denotes the complement in $X_j^k \mid V_j$ of the smooth components of the latter on which $\mathrm{inv}^e_{\underline{X}}$ is constant. (These components are necessarily open and closed.) We obtain a finite sequence of blowings-up of $X^0|V$ with smooth $\mathrm{inv}_{\underline{X}}$-admissible centres, satisfying the conditions of the lemma for $\underline{X}|V$. Again we want to extend the successive centres to global analytic subspaces and resolve their singularities (which lie outside the V_j) by the inductive assumption.

Suppose we have a locally finite sequence (13.6) of $\mathrm{inv}_{\underline{X}}$-admissible blowings-up of X^0 satisfying (1) and (3) of the lemma, and restricting over V to part of the resolution tower for $\underline{X}|V$. Write $X^i_\bullet := \varprojlim X^i_j$, $i = 0, \ldots, k$. (In particular, the induced morphism $\sigma_\bullet \colon X^0_\bullet \to X^0$ is the composite of the sequence of blowings-up.)

Set $W = \sigma_\bullet^{-1}(V)$. Let Λ denote the (finite) set of maximal values $\lambda = \mathrm{inv}^e_{\underline{X}}(a)$ of $\mathrm{inv}^e_{\underline{X}}$ on $Y^k_\bullet$ (where $Y^k_\bullet$ denotes the complement in $X^k_\bullet \mid W$ of the smooth components of the latter on which $\mathrm{inv}^e_{\underline{X}}$ is constant), and let $S := \bigcup_{\lambda\in\Lambda}\{x \in Y^k_\bullet : \mathrm{inv}^e_{\underline{X}}(a) = \lambda\}$. Then S defines a smooth closed subspace of $X^k_\bullet|W$ and $\dim S < \dim X^k_\bullet$. For each $\lambda \in \Lambda$, let T_λ denote the smallest closed analytic subspace of $X^k_\bullet$ such that $|T_\lambda| = \{x \in X^k_\bullet : \mathrm{inv}^e_{\underline{X}}(x) \geq \lambda\}$. Let S_q denote the union of the components of S of the smallest dimension q. Let $X^{k+1}_\bullet$ denote the union of the components of $\bigcup_{\lambda\in\Lambda} T_\lambda$ of dimension q. Then $X^{k+1}_\bullet|W = S_q$ and $\bigcup_{\lambda\in\Lambda}\{x \in |X^{k+1}_\bullet| : \mathrm{inv}^e_{\underline{X}}(x) = \lambda\}$ is Zariski-open in $|X^{k+1}_\bullet|$.

By induction on the dimension of the last space $X^{k+1}_\bullet$, we can assume that Lemma 13.7 holds for $X^0_\bullet \supset X^1_\bullet \supset \cdots \supset X^{k+1}_\bullet$. The sequence of blowings-up involved will be trivial over W, and the final transform of $X^{k+1}_\bullet$ will be a smooth extension of S above on which ι^k is locally constant. □

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